1) a) $\vec{B} = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}+\vec{\nabla}f) = \vec{\nabla}\times\vec{A} + \overbrace{\cancel{\vec{\nabla}\times\vec{\nabla}f}}^{0} = \vec{\nabla}\times\vec{A}$ ✓

(Because $\vec{\nabla}\times\vec{\nabla}f = 0$ for any function f.)

b) $\vec{E} = -\vec{\nabla}\psi' - \frac{\partial}{\partial t}\vec{A}' = -\vec{\nabla}(\psi - \frac{\partial f}{\partial t}) - \frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}f) = -\vec{\nabla}\psi - \frac{\partial}{\partial t}\vec{A} + \cancel{\frac{\partial}{\partial t}\vec{\nabla}f} - \cancel{\frac{\partial}{\partial t}\vec{\nabla}f}$

$\Rightarrow \vec{E} = -\vec{\nabla}\psi - \frac{\partial}{\partial t}\vec{A}$ ✓

c) $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\psi'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}f + \frac{1}{c^2}\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \overbrace{\cancel{(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\psi}{\partial t})}}^{0} + (\vec{\nabla}^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}) = 0$

$\Rightarrow f(\vec{r},t)$ must be a solution of the second-order differential equation $\vec{\nabla}^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$. ✓

d) $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}f = 0 \Rightarrow \vec{\nabla}^2 f = -\vec{\nabla}\cdot\vec{A} = \frac{1}{c^2}\frac{\partial\psi}{\partial t}$. ✓

Note that, in the Lorentz gauge, $\vec{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\rho/\varepsilon_0 \Rightarrow \vec{\nabla}^2\psi - \frac{\partial}{\partial t}\vec{\nabla}^2 f = -\rho/\varepsilon_0$

$\Rightarrow \vec{\nabla}^2(\psi - \frac{\partial f}{\partial t}) = -\rho/\varepsilon_0 \Rightarrow \vec{\nabla}^2\psi' = -\rho/\varepsilon_0$. Thus, in the Coulomb gauge, the scalar potential $\psi'(\vec{r},t)$ is obtained from the charge density distribution $\rho(\vec{r},t)$ in exactly the same way as in electrostatics. In other words, the delay $|\vec{r}-\vec{r}'|/c$ due to the finite travel time between $\vec{r}$ and $\vec{r}'$ is ignored. As for the vector potential $\vec{A}'(\vec{r},t)$, we find the following differential equation starting with Maxwell's 2nd equation (assuming $\vec{P}=0$, $\vec{M}=0$):

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}') = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}(-\vec{\nabla}\psi' - \frac{\partial\vec{A}'}{\partial t})$

$\Rightarrow \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}'}}_{0}) - \vec{\nabla}^2\vec{A}' = \mu_0\vec{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\psi' - \frac{1}{c^2}\frac{\partial^2\vec{A}'}{\partial t^2} \Rightarrow \vec{\nabla}^2\vec{A}' - \frac{1}{c^2}\frac{\partial^2\vec{A}'}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\psi'$. ✓

The second-order partial differential equation for $\vec{A}'$ thus involves as source terms not only $\vec{J}$ but also $\frac{\partial}{\partial t}\vec{\nabla}\psi'$, which is derived from the charge distribution $\rho(\vec{r},t)$.

2) a) Denoting by "a" the cross-sectional area of the wire, we find $\vec{J}(z,t) = (I_0/a)\sin(\omega t - kz)\hat{z}$. The continuity equation $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$ thus yields: $\frac{\partial\rho(z,t)}{\partial t} = -\frac{\partial J_z(z,t)}{\partial z} = (kI_0/a)\cos(\omega t - kz) \Rightarrow$

$\rho(z,t) = \frac{kI_0}{\omega a}\sin(\omega t - kz) \Rightarrow$ linear charge density $\lambda(z,t) = a\rho(z,t) = \frac{kI_0}{\omega}\sin(\omega t - kz)$. ✓

b) $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\psi}{\partial t} = \frac{\partial A_z}{\partial z} + \mu_0\epsilon_0\frac{\partial\psi}{\partial t} = -\frac{\mu_0 I_0}{4}\left\{-\kappa Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) + \kappa J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\} - \frac{\mu_0\kappa I_0}{4\omega}\left\{\omega Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - \omega J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\} = 0$ ✓ ← Lorentz Gauge

c) $\vec{E}(\rho,z,t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = -\frac{\partial\psi}{\partial\rho}\hat{\rho} - \left(\frac{\partial\psi}{\partial z} + \frac{\partial A_z}{\partial t}\right)\hat{z} \Rightarrow$

$$E_\rho(\rho,z,t) = -\frac{\kappa I_0\sqrt{k_0^2-\kappa^2}}{4\epsilon_0\omega}\left\{Y_1(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z)\right\} \checkmark$$

$$E_z(\rho,z,t) = \left(-\frac{\kappa^2 I_0}{4\epsilon_0\omega} + \frac{\mu_0 I_0\omega}{4}\right)\left\{Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\}$$

$$= \frac{(k_0^2-\kappa^2)I_0}{4\epsilon_0\omega}\left\{Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\} \checkmark$$

$$\vec{H}(\rho,z,t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = -\frac{1}{\mu_0}\frac{\partial A_z}{\partial\rho}\hat{\phi} \Rightarrow$$

$$H_\phi(\rho,z,t) = -\frac{1}{4}I_0\sqrt{k_0^2-\kappa^2}\left\{Y_1(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z)\right\} \checkmark$$

d) In the limit of large ρ we'll have: $J_n(x) \to \sqrt{\frac{2}{\pi x}}\cos(x - \frac{n\pi}{2} - \frac{\pi}{4})$ and $Y_n(x) \to \sqrt{\frac{2}{\pi x}}\sin(x - \frac{n\pi}{2} - \frac{\pi}{4})$. Therefore,

$$E_\rho(\rho,z,t) \to -\frac{\kappa I_0\sqrt{k_0^2-\kappa^2}}{4\epsilon_0\omega}\sqrt{\frac{2}{\pi\sqrt{k_0^2-\kappa^2}\rho}}\left\{\sin(\sqrt{k_0^2-\kappa^2}\rho - \tfrac{3\pi}{4})\sin(\omega t-\kappa z) + \cos(\sqrt{k_0^2-\kappa^2}\rho - \tfrac{3\pi}{4})\cos(\omega t-\kappa z)\right\} = \frac{\kappa I_0\sqrt[4]{k_0^2-\kappa^2}}{2\epsilon_0\omega\sqrt{2\pi\rho}}\cos(\sqrt{k_0^2-\kappa^2}\rho + \kappa z - \omega t + \tfrac{\pi}{4}) \checkmark$$

$$E_z(\rho,z,t) \to \frac{-I_0(k_0^2-\kappa^2)^{3/4}}{2\epsilon_0\omega\sqrt{2\pi\rho}}\cos(\sqrt{k_0^2-\kappa^2}\rho + \kappa z - \omega t + \pi/4) \checkmark$$

$$H_\phi(\rho,z,t) \to \frac{+I_0\sqrt[4]{k_0^2-\kappa^2}}{2\sqrt{2\pi\rho}}\cos(\sqrt{k_0^2-\kappa^2}\rho + \kappa z - \omega t + \pi/4) \checkmark$$

$$\vec{S}(\rho,z,t) = (E_\rho\hat{\rho} + E_z\hat{z})\times H_\phi\hat{\phi} = -E_zH_\phi\hat{\rho} + E_\rho H_\phi\hat{z}$$

For $\rho\to\infty$ we'll have $\vec{S}(\rho,z,t) \to \frac{\sqrt{k_0^2-\kappa^2}\,I_0^2}{8\pi\epsilon_0\omega\rho}(\sqrt{k_0^2-\kappa^2}\,\hat{\rho} + \kappa\hat{z})\cos^2(\sqrt{k_0^2-\kappa^2}\rho + \kappa z - \omega t + \pi/4)$

3) a) $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\psi}{\partial t} = 0 \Rightarrow ik_0\vec{\sigma}\cdot\vec{A}_0 - \frac{i\omega}{c^2}\psi_0 = 0 \Rightarrow \vec{\sigma}\cdot\vec{A}_0 = \psi_0/c$ ✓

We have used the relation $k_0 = \omega/c$ in arriving at the above formula.

b) $\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = -(ik_0\psi_0\vec{\sigma} - i\omega\vec{A}_0)e^{i(k_0\vec{\sigma}\cdot\vec{r}-\omega t)} \Rightarrow$

$\vec{E}(\vec{r},t) = i\omega(\vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma})\exp[i(k_0\vec{\sigma}\cdot\vec{r}-\omega t)]$ ✓

$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\, ik_0\vec{\sigma}\times\vec{A}_0 \exp[i(k_0\vec{\sigma}\cdot\vec{r}-\omega t)]$ ✓

c) ① $\vec{\nabla}\cdot\vec{D} = \rho_{free} \Rightarrow \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\sigma}\cdot\vec{E} = 0 \Rightarrow \vec{\sigma}\cdot(\vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma}) = 0 \Rightarrow$

$\vec{\sigma}\cdot\vec{A}_0 = \frac{\psi_0}{c}\vec{\sigma}\cdot\vec{\sigma} \Rightarrow \vec{\sigma}\cdot\vec{A}_0 = \frac{\psi_0}{c}$ ✓ This is satisfied because of the Lorentz gauge.

② $\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t} \Rightarrow (1/\mu_0)(ik_0)^2\vec{\sigma}\times(\vec{\sigma}\times\vec{A}_0) = -i\omega\epsilon_0[i\omega(\vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma})] \Rightarrow$

$-\frac{k_0^2}{\mu_0}[(\vec{\sigma}\cdot\vec{A}_0)\vec{\sigma} - (\vec{\sigma}\cdot\vec{\sigma})\vec{A}_0] = \omega^2\epsilon_0(\vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma}) \xrightarrow[k_0=\omega/c]{\mu_0\epsilon_0=1/c^2} (\vec{\sigma}\cdot\vec{\sigma})\vec{A}_0 - (\vec{\sigma}\cdot\vec{A}_0)\vec{\sigma} = \vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma}$

But $\vec{\sigma}\cdot\vec{\sigma} = 1$ and $\vec{\sigma}\cdot\vec{A}_0 = \psi_0/c$ (Lorentz gauge). Therefore Maxwell's 2nd is satisfied.

③ $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow ik_0\vec{\sigma}\times[i\omega(\vec{A}_0 - \frac{\psi_0}{c}\vec{\sigma})] = i\omega\mu_0[\frac{1}{\mu_0}ik_0\vec{\sigma}\times\vec{A}_0] \Rightarrow$

$i^2\cancel{k_0\omega}(\vec{\sigma}\times\vec{A}_0 - \frac{\psi_0}{c}\underbrace{\cancel{\vec{\sigma}\times\vec{\sigma}}}_{0}) = i^2\cancel{k_0\omega}\,\vec{\sigma}\times\vec{A}_0 \Rightarrow \vec{\sigma}\times\vec{A}_0 = \vec{\sigma}\times\vec{A}_0$ ✓

④ $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow ik_0\vec{\sigma}\cdot(ik_0\vec{\sigma}\times\vec{A}_0) = 0 \Rightarrow \vec{\sigma}\cdot(\vec{\sigma}\times\vec{A}_0) = (\underbrace{\cancel{\vec{\sigma}\times\vec{\sigma}}}_{0})\cdot\vec{A}_0 = 0$ ✓

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4) a) $\vec{\sigma}\cdot\vec{\sigma} = 1 \Rightarrow (i\sigma_x\hat{x} + \sigma_z\hat{z})\cdot(i\sigma_x\hat{x} + \sigma_z\hat{z}) = -\sigma_x^2 + \sigma_z^2 = 1 \Rightarrow \sigma_z^2 = 1 + \sigma_x^2$ ✓

b) $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\sigma}\cdot\vec{E}_0 = 0 \Rightarrow i\sigma_x E_{x0} + \sigma_z E_{z0} = 0$ ✓

c) $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\sigma}\cdot\vec{H}_0 = 0 \Rightarrow i\sigma_x H_{x0} + \sigma_z H_{z0} = 0$ ✓

d) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \cancel{i}k_0\vec{\sigma}\times\vec{E}_0 = \cancel{i}\omega\mu_0\vec{H}_0 \Rightarrow (i\sigma_x\hat{x} + \sigma_z\hat{z})\times(E_{x0}\hat{x} + E_{y0}\hat{y} + E_{z0}\hat{z}) = Z_0\vec{H}_0$

$\Rightarrow i\sigma_x E_{y0}\hat{z} + (\sigma_z E_{x0} - i\sigma_x E_{z0})\hat{y} - \sigma_z E_{y0}\hat{x} = Z_0\vec{H}_0 \Rightarrow$

$Z_0 H_{0x} = -\sigma_z E_{y0}$; $Z_0 H_{0y} = \sigma_z E_{x0} - i\sigma_x E_{z0}$; $Z_0 H_{0z} = i\sigma_x E_{y0}$ ✓

e) If $E_{z0} = 0$, then from (b) we have $E_{x0} = 0$ and from (d) we find $H_{y0} = 0$. ✓

f) If $H_{z0} = 0$, then from (c) we have $H_{x0} = 0$ and from (d) we find $E_{y0} = 0$. ✓