1) a) No. At $t=0$ we have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}$, while at $t=T/4$, where T is the period of the oscillation, we'll have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0I}$. If $\vec{E}_{0R}$ and $\vec{E}_{0I}$ are not aligned, the $\vec{E}$-field will have two different orientations at $t=0$ and $t=T/4$. For linear polarization, therefore, it is necessary for $\vec{E}_{0R}$ and $\vec{E}_{0I}$ to be aligned.

b) $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}\cos\omega t+\vec{E}_{0I}\sin\omega t$

When $\vec{E}_{0R}$ and $\vec{E}_{0I}$ are aligned their common direction will be the direction of $\vec{E}(\vec{r}_0,t)$. The length of $\vec{E}(\vec{r}_0,t)$ can then be written in terms of the lengths of the vectors $\vec{E}_{0R}$ and $\vec{E}_{0I}$, namely, E_{0R} and E_{0I}, as follows:

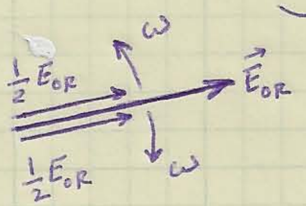
$$E(\vec{r}_0,t)=E_{0R}\cos\omega t+E_{0I}\sin\omega t=\sqrt{E_{0R}^2+E_{0I}^2}\Big(\underbrace{\frac{E_{0R}}{\sqrt{E_{0R}^2+E_{0I}^2}}}_{\cos\phi}\cos\omega t+\underbrace{\frac{E_{0I}}{\sqrt{E_{0R}^2+E_{0I}^2}}}_{\sin\phi}\sin\omega t\Big)$$

$$=\sqrt{E_{0R}^2+E_{0I}^2}\,(\cos\phi\cos\omega t+\sin\phi\sin\omega t)=\sqrt{E_{0R}^2+E_{0I}^2}\,\cos(\omega t-\phi)$$

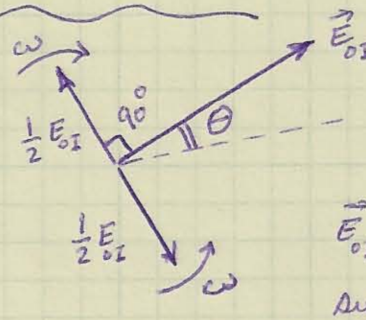
The magnitude of the $\vec{E}$-field is, therefore, $\sqrt{E_{0R}^2+E_{0I}^2}$.

c) No. At $t=0$ we have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}$, while at $t=T/4$ we'll have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0I}$. A circularly-polarized beam must have the same E-field magnitude at all times. Therefore, if $|\vec{E}_{0R}|\neq|\vec{E}_{0I}|$, the E-field magnitude at $t=0$ will differ from its magnitude at $t=T/4$, which means that the beam is not circularly polarized.

d)



$\vec{E}_{0R}\cos\omega t$ is the superposition of two circularly-polarized beams



$\vec{E}_{0I}\sin\omega t$ is also the superposition of two circularly-polarized beams.

In general, the two RCP beams combine to produce a single RCP beam. Similarly, the two LCP beams combine to produce a single LCP beam. When $\theta = \pm 90°$, however, either the two RCP beams cancel each other out, or the two LCP beams cancel each other out. The result will then be a single circularly polarized beam. When $\theta \neq \pm 90°$, both RCP and LCP beams will have non-zero magnitudes, and the resulting beam can't be a pure circularly-polarized beam.

e) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow i k_0 \vec{\sigma} \times \vec{E}_0 = -(-i\omega)\mu_0 \vec{H}_0 \Rightarrow i\frac{\omega}{c}\vec{\sigma} \times \vec{E}_0 = i\omega\mu_0 \vec{H}_0$

$\Rightarrow \vec{\sigma} \times \vec{E}_0 = Z_0 \vec{H}_0.$

For a homogeneous plane-wave $\vec{\sigma} = \vec{\sigma}_R$; therefore,

$$Z_0 \vec{H}_0 = \vec{\sigma}_R \times (\vec{E}_{0R} + i\vec{E}_{0I}) \Rightarrow \begin{cases} Z_0 \vec{H}_{0R} = \vec{\sigma}_R \times \vec{E}_{0R} \\ Z_0 \vec{H}_{0I} = \vec{\sigma}_R \times \vec{E}_{0I} \end{cases}$$

From Maxwell's first equation, $\vec{\nabla} \cdot \vec{E} = 0$, we know that $\vec{\sigma} \cdot \vec{E} = 0$. Therefore, for a homogeneous plane-wave, $\vec{\sigma}_R \cdot (\vec{E}_{0R} + i\vec{E}_{0I}) = 0$

$\Rightarrow \vec{\sigma}_R \cdot \vec{E}_{0R} = 0$ and $\vec{\sigma}_R \cdot \vec{E}_{0I} = 0$. In other words, the unit-vector $\vec{\sigma}_R$ is perpendicular to both $\vec{E}_{0R}$ and $\vec{E}_{0I}$. We conclude that $\vec{\sigma}_R$, $\vec{E}_{0R}$ and $\vec{H}_{0R}$ are mutually orthogonal, and also $\vec{\sigma}_R$, $\vec{E}_{0I}$ and $\vec{H}_{0I}$ are mutually orthogonal. The magnitudes of $\vec{H}_{0R}$ and $\vec{H}_{0I}$ are thus given by: $|\vec{H}_{0R}| = |\vec{E}_{0R}|/Z_0$ and $|\vec{H}_{0I}| = |\vec{E}_{0I}|/Z_0$.

f) $\vec{E}(\vec{r},t) = Re\{\vec{E}_0 \exp[i(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)]\} = \vec{E}_{0R}\cos(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t) - \vec{E}_{0I}\sin(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)$

$\vec{H}(\vec{r},t) = \vec{H}_{0R}\cos(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t) - \vec{H}_{0I}\sin(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)$

$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \vec{E}_{0R} \times \vec{H}_{0R}\cos^2(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t) + \vec{E}_{0I} \times \vec{H}_{0I}\sin^2(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)$

$- (\vec{E}_{0R} \times \vec{H}_{0I} + \vec{E}_{0I} \times \vec{H}_{0R})\sin(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)\cos(k_0\vec{\sigma}_R \cdot \vec{r} - \omega t)$

Using the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$, we write:

$$\vec{E}_{0R}\times\vec{H}_{0R} = \frac{1}{Z_0}\vec{E}_{0R}\times(\vec{\sigma}_R\times\vec{E}_{0R}) = \frac{1}{Z_0}(\vec{E}_{0R}\cdot\vec{E}_{0R})\vec{\sigma}_R - \frac{1}{Z_0}\overset{0}{\cancel{(\vec{E}_{0R}\cdot\vec{\sigma}_R)}}\vec{E}_{0R}$$

$$\vec{E}_{0I}\times\vec{H}_{0I} = \frac{1}{Z_0}\vec{E}_{0I}\times(\vec{\sigma}_R\times\vec{E}_{0I}) = \frac{1}{Z_0}(\vec{E}_{0I}\cdot\vec{E}_{0I})\vec{\sigma}_R - \frac{1}{Z_0}\overset{0}{\cancel{(\vec{E}_{0I}\cdot\vec{\sigma}_R)}}\vec{E}_{0I}$$

$$\vec{E}_{0R}\times\vec{H}_{0I} = \frac{1}{Z_0}\vec{E}_{0R}\times(\vec{\sigma}_R\times\vec{E}_{0I}) = \frac{1}{Z_0}(\vec{E}_{0R}\cdot\vec{E}_{0I})\vec{\sigma}_R - \frac{1}{Z_0}\overset{0}{\cancel{(\vec{E}_{0R}\cdot\vec{\sigma}_R)}}\vec{E}_{0I}$$

$$\vec{E}_{0I}\times\vec{H}_{0R} = \frac{1}{Z_0}\vec{E}_{0I}\times(\vec{\sigma}_R\times\vec{E}_{0R}) = \frac{1}{Z_0}(\vec{E}_{0I}\cdot\vec{E}_{0R})\vec{\sigma}_R - \frac{1}{Z_0}\overset{0}{\cancel{(\vec{E}_{0I}\cdot\vec{\sigma}_R)}}\vec{E}_{0R}$$

Therefore,

$$\vec{S}(\vec{r},t) = \frac{\vec{\sigma}_R}{Z_0}\Big\{\vec{E}_{0R}\cdot\vec{E}_{0R}\cos^2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t) + \vec{E}_{0I}\cdot\vec{E}_{0I}\sin^2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t) - \vec{E}_{0R}\cdot\vec{E}_{0I}\sin\left[2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t)\right]\Big\} \Rightarrow$$

$$\vec{S}(\vec{r},t) = \frac{\vec{\sigma}_R}{2Z_0}\Big\{(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2) + (|\vec{E}_{0R}|^2-|\vec{E}_{0I}|^2)\cos(2k_0\vec{\sigma}_R\cdot\vec{r}-2\omega t) - 2\vec{E}_{0R}\cdot\vec{E}_{0I}\sin(2k_0\vec{\sigma}_R\cdot\vec{r}-2\omega t)\Big\}$$

Note that, for circularly polarized beams, $|\vec{E}_{0R}| = |\vec{E}_{0I}|$ and $\vec{E}_{0R}\cdot\vec{E}_{0I} = 0$; therefore, $\vec{S}(\vec{r},t) = \langle\vec{S}(\vec{r},t)\rangle$, that is, the rate of flow of energy is independent of $\vec{r}$ and t.

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2) a) $$\begin{cases}\vec{E}(z,t) = E_{x0}\cos(k_0 z-\omega t+\phi_0)\,\hat{x}\\ \vec{H}(z,t) = \frac{1}{Z_0}E_{x0}\cos(k_0 z-\omega t+\phi_0)\,\hat{y}\end{cases}$$ ← $\vec{E}$ and $\vec{H}$ have the same phase, ϕ_0.

$$\begin{cases}\vec{E}'(z,t) = E_{x0}\cos(k_0 z+\omega t+\phi_0')\,\hat{x}\\ \vec{H}'(z,t) = \frac{-1}{Z_0}E_{x0}\cos(k_0 z+\omega t+\phi_0')\,\hat{y}\end{cases}$$ ← $\vec{E}'$ and $\vec{H}'$ have the same phase, ϕ_0'.

Total E-field $= \vec{E}(z,t)+\vec{E}'(z,t) = E_{x0}\left[\cos(k_0 z-\omega t+\phi_0)+\cos(k_0 z+\omega t+\phi_0')\right]\hat{x}$

$$= 2E_{x0}\cos\left(k_0 z+\frac{\phi_0+\phi_0'}{2}\right)\cos\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{x}$$

Total H-field $= \vec{H}(z,t)+\vec{H}'(z,t) = 2\,\frac{E_{x0}}{Z_0}\sin\left(k_0 z+\frac{\phi_0+\phi_0'}{2}\right)\sin\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{y}$

Since the total E-field at the mirror surfaces must be zero, we'll have

$$\begin{cases} \text{First mirror at } z=0: & 2E_{x_0}\cos\left(\frac{\phi_0+\phi_0'}{2}\right)\cos\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{x}=0 \\ \text{Second mirror at } z=d: & 2E_{x_0}\cos\left(k_0 d+\frac{\phi_0+\phi_0'}{2}\right)\cos\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{x}=0 \end{cases} \Rightarrow$$

$$\begin{cases} \cos\left(\frac{\phi_0+\phi_0'}{2}\right)=0 \\ \cos\left(k_0 d+\frac{\phi_0+\phi_0'}{2}\right)=0 \end{cases} \Rightarrow \begin{cases} \phi_0+\phi_0'=(2n+1)\pi \leftarrow \text{odd multiple of } \pi \\ k_0 d = m\pi \Rightarrow d = m\lambda_0/2 \leftarrow \text{Integer multiple of } \lambda_0/2 \end{cases} \Rightarrow$$

$$\boxed{\begin{aligned} &\text{Total E-field} = 2E_{x_0}\sin(k_0 z)\cos\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{x} \\ &\text{Total H-field} = -\frac{2E_{x_0}}{Z_0}\cos(k_0 z)\sin\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)\hat{y} \end{aligned}} \leftarrow \text{Cavity fields}$$

b) The surface current density is equal to the total H-field at the mirror surfaces. For the first mirror $z=0 \Rightarrow \cos(k_0 z)=1$. For the second mirror $z=d \Rightarrow \cos(k_0 z)=\cos(m\pi)=\pm 1$. The magnitude of the surface current on both mirrors, therefore, is $J_{s_0}=2E_{x_0}/Z_0$.

c) The trapped energy per <u>unit</u> cross-sectional area is given by:

$$\text{Trapped E-field energy} = \frac{1}{2}\epsilon_0\int_0^d |\vec{E}(z,t)|^2 dz = 2\epsilon_0 E_{x_0}^2\left(\int_0^d \sin^2(k_0 z)dz\right)\cos^2\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)$$

$$= (\epsilon_0 E_{x_0}^2 d)\cos^2\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)$$

$$\text{Trapped H-field energy} = \frac{1}{2}\mu_0\int_0^d |\vec{H}(z,t)|^2 dz = 2\mu_0\frac{E_{x_0}^2}{Z_0^2}\left(\int_0^d \cos^2(k_0 z)dz\right)\sin^2\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)$$

$$= (\epsilon_0 E_{x_0}^2 d)\sin^2\left(\omega t+\frac{\phi_0'-\phi_0}{2}\right)$$

The maximum E-field and H-field energies are thus equal to $\epsilon_0 E_{x_0}^2 d$. However, there exists a phase difference between these two entities. When the E-field energy is zero, the H-field energy is at a maximum and vice-versa.

At one instant, all the energy is in the E-field. A quarter of a period later, all the energy is in the H-field. The energy thus swings back and forth from one form to another.

d) $$\vec{S}(z,t) = \vec{E}(z,t) \times \vec{H}(z,t) = -\frac{4E_{x0}^2 \hat{z}}{Z_0} \sin(k_0 z)\cos(k_0 z)\sin\left(\omega t + \frac{\phi_0' - \phi_0}{2}\right)\cos\left(\omega t + \frac{\phi_0' - \phi_0}{2}\right)$$

$$\Rightarrow \vec{S}(z,t) = -\frac{E_{x0}^2 \hat{z}}{Z_0} \sin(2k_0 z)\sin(2\omega t + \phi_0' - \phi_0)$$

At the nodes of the E-field, as well as those of the H-field, the Poynting Vector is zero. No energy, therefore, crosses these nodes. In between the nodes, the energy flows to the right for one quarter of one period (T/4), then flows to the left during the next quarter. The process is then repeated.

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3) a) In the region between the two cylinders, $R_1 < \rho < R_2$, we'll have:

$$\vec{E}(\rho,t) = \frac{1}{4} k_0 Z_0 \hat{z} \left\{ I_{10} J_0(k_0 R_1)\left[Y_0(k_0\rho)\cos\omega t - J_0(k_0\rho)\sin\omega t\right] + I_{20}\left[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\right] J_0(k_0\rho)\right\}$$

Boundary Condition: $\vec{E}(R_1,t) = \vec{E}(R_2,t) = 0 \Rightarrow$

$$\begin{cases} R_2 \rightarrow & I_{10} J_0(k_0R_1)\left[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\right] + I_{20}\left[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\right] J_0(k_0R_2) = 0 \\ R_1 \rightarrow & I_{10} J_0(k_0R_1)\left[Y_0(k_0R_1)\cos\omega t - J_0(k_0R_1)\sin\omega t\right] + I_{20}\left[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\right] J_0(k_0R_1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left[I_{10}J_0(k_0R_1) + I_{20}J_0(k_0R_2)\right]\left[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\right] = 0 \\ \left\{\left[I_{10}Y_0(k_0R_1) + I_{20}Y_0(k_0R_2)\right]\cos\omega t - \left[I_{10}J_0(k_0R_1) + I_{20}J_0(k_0R_2)\right]\sin\omega t\right\} J_0(k_0R_1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} I_{10}J_0(k_0R_1) + I_{20}J_0(k_0R_2) = 0 \\ I_{10}Y_0(k_0R_1) + I_{20}Y_0(k_0R_2) = 0 \end{cases} \Rightarrow \begin{cases} I_{10}/I_{20} = -J_0(k_0R_2)/J_0(k_0R_1) \\ I_{10}/I_{20} = -Y_0(k_0R_2)/Y_0(k_0R_1) \end{cases}$$

A standing wave will thus exist in the cavity between the two cylinders if the following condition is satisfied:

$$\frac{J_0(k_0R_2)}{J_0(k_0R_1)} = \frac{Y_0(k_0R_2)}{Y_0(k_0R_1)}$$

As for the magnetic field within the cavity between the two cylinders, we have

$$\vec{H}(\rho,t) = -\frac{1}{4} k_0 \hat{\Phi} \Big\{ I_{10} J_0(k_0R_1)\left[Y_1(k_0\rho)\sin\omega t + J_1(k_0\rho)\cos\omega t\right] +$$

$$I_{20}\left[Y_0(k_0R_2)\sin\omega t + J_0(k_0R_2)\cos\omega t\right] J_1(k_0\rho)\Big\}$$

The magnetic field at the cylinder surfaces will be:

$$\vec{H}(R_1,t) = -\frac{1}{4} k_0 \hat{\Phi} \Big\{ \left[I_{10} J_0(k_0R_1) Y_1(k_0R_1) + I_{20} J_1(k_0R_1) Y_0(k_0R_2)\right]\sin\omega t +$$

$$\cancelto{0}{\left[I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2)\right]} J_1(k_0R_1)\cos\omega t\Big\} \Rightarrow$$

$$\vec{H}(R_1,t) = -\frac{1}{4} k_0 I_{10} \hat{\Phi} \left[J_0(k_0R_1)Y_1(k_0R_1) - J_1(k_0R_1)Y_0(k_0R_1)\right]\sin(\omega t)$$

$$= \frac{1}{4} k_0 I_{10} \hat{\Phi} \left(\frac{2}{\pi k_0 R_1}\right)\sin(\omega t) = \frac{I_{10}\sin(\omega t)}{2\pi R_1}\hat{\Phi} = J_{s1}(t)\hat{\Phi}$$

$$\vec{H}(R_2,t) = -\frac{1}{4} k_0 \hat{\Phi} \Big\{ \left[I_{10} J_0(k_0R_1) Y_1(k_0R_2) + I_{20} Y_0(k_0R_2) J_1(k_0R_2)\right]\sin(\omega t) +$$

$$\cancelto{0}{\left[I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2)\right]} J_1(k_0R_2)\cos(\omega t)\Big\} \Rightarrow$$

$$\vec{H}(R_2,t) = -\frac{1}{4} k_0 \hat{\Phi} I_{20} \left[-J_0(k_0R_2)Y_1(k_0R_2) + Y_0(k_0R_2)J_1(k_0R_2)\right]\sin\omega t$$

$$= -\frac{1}{4} k_0 I_{20} \hat{\Phi} \left(\frac{2}{\pi k_0 R_2}\right)\sin\omega t = -\frac{I_{20}\sin(\omega t)}{2\pi R_2}\hat{\Phi} = -J_{s2}(t)\hat{\Phi}$$

Clearly the H-field on the surface of the conductors is equal to $J_s(t)$ and perpendicular to the direction of the current.

b) Inside the small cylinder $\rho < R_1 < R_2$. The fields are given by:

$$\vec{E}(\rho,t) = \frac{1}{4} k_0 Z_0 \hat{z} \Big\{ I_{10}\big[Y_0(k_0R_1)\cos\omega t - J_0(k_0R_1)\sin\omega t\big] J_0(k_0\rho) +$$

$$I_{20}\big[Y_0(k_0R_2)\cos\omega t - J_0(k_0R_2)\sin\omega t\big] J_0(k_0\rho) \Big\}$$

$$= \frac{1}{4} k_0 Z_0 \hat{z} \Big\{ \cancelto{0}{\big[I_{10} Y_0(k_0R_1) + I_{20} Y_0(k_0R_2)\big]} \cos\omega t - \cancelto{0}{\big[I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2)\big]} \sin\omega t \Big\} J_0(k_0\rho)$$

$$= 0 \quad ✓$$

$$\vec{H}(\rho,t) = -\frac{1}{4} k_0 \hat{\phi} J_1(k_0\rho) \Big\{ \cancelto{0}{\big[I_{10} Y_0(k_0R_1) + I_{20} Y_0(k_0R_2)\big]} \sin\omega t + \cancelto{0}{\big[I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2)\big]} \cos\omega t \Big\}$$

$$= 0 \quad ✓$$

Outside the large cylinder $\rho > R_2 > R_1$. The fields are given by:

$$\vec{E}(\rho,t) = \frac{1}{4} k_0 Z_0 \hat{z} \cancelto{0}{\big\{ I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2) \big\}} \big[Y_0(k_0\rho)\cos\omega t - J_0(k_0\rho)\sin\omega t\big] = 0 \quad ✓$$

$$\vec{H}(\rho,t) = -\frac{1}{4} k_0 \hat{\phi} \big[Y_1(k_0\rho)\sin\omega t + J_1(k_0\rho)\cos\omega t\big] \cancelto{0}{\big[I_{10} J_0(k_0R_1) + I_{20} J_0(k_0R_2)\big]} = 0 \quad ✓$$

Thus the fields vanish from the inside of the small cylinder as well as the outside of the large cylinder. The entire radiation is then confined to the region between the two cylinders. This problem is the analogue of the preceding problem, where the cavity was the region between two plane-parallel mirrors. The resonance condition in the case of plane-parallel mirrors was that the distance between mirrors must be an integer-multiple of $\lambda_0/2$. Here the resonance condition is the relation $J_0(k_0R_2)/J_0(k_0R_1) = Y_0(k_0R_2)/Y_0(k_0R_1)$. Only those values of R_1 and R_2 that satisfy the resonance condition can trap an electromagnetic field between the two cylinders. Note that the ratio of the currents, I_{10}/I_{20}, which is equal to $-J_0(k_0R_2)/J_0(k_0R_1)$ can be either positive or negative, depending on the values of R_1, R_2, λ_0.