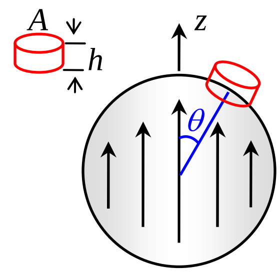
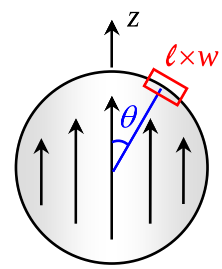
Problem 1) From Maxwell's 4th equation, we find the bound magnetic charge-density to be given by $\rho_{\text{bound}}^{(m)} = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r})$. Take a small pillbox and place it anywhere inside the sphere. The magnetization entering the box will be equal to that leaving the box and, therefore, the divergence of $\boldsymbol{M}(\boldsymbol{r}) = M_o\hat{\boldsymbol{z}}$ will be zero everywhere inside the sphere. The only points where the divergence will be non-zero are at the surface of the sphere. The figure shows a small, thin pillbox placed at $(r=R,\theta,\phi)$. Let A and h denote the base area and height of the pillbox, respectively; both A and h could be as small as desired. The flux of $\boldsymbol{M}$ entering from the bottom of the pillbox is $M_o A\cos\theta$, and this is the only contribution to the integral of $\boldsymbol{M}(\boldsymbol{r})$ over the pillbox surface, provided that h is much small than the pillbox diameter. The divergence of $\boldsymbol{M}$ at $(r=R,\theta,\phi)$ is thus given by $-M_o A\cos\theta/(Ah)$ in the limit of small A and h. Therefore, $\rho_{\text{bound}}^{(m)}(R,\theta,\phi) = M_o\cos\theta/h$. Since the charges are confined to the surface, we should use the surface-charge-density $\sigma_{\text{bound}}^{(m)} = h\rho_{\text{bound}}^{(m)}$ instead of the volume charge-density. Consequently, $\sigma_{\text{bound}}^{(m)}(R,\theta,\phi) = M_o\cos\theta$.



To determine the bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(e)} = \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r})$, we use a small rectangular loop (length $=\ell$, width $=w$) at various locations within and on the surface of the sphere in order to calculate the curl of $\boldsymbol{M}$. For all locations within the sphere and for all orientations of the loop, the integral of $\boldsymbol{M}(\boldsymbol{r})$ around the loop turns out to be zero. When the loop is placed on the surface at $(r=R,\theta,\phi)$ and oriented perpendicular to $\hat{\boldsymbol{\phi}}$, as shown, the line integral on the lower leg of the loop will be nonzero $(\ell M_o\sin\theta)$. The curl of $\boldsymbol{M}(\boldsymbol{r})$ will then be nonzero, as the other legs do not contribute to the integral, provided that $w \ll \ell$. The curl will then be given by $[\ell M_o\sin\theta/(\ell w)]\hat{\boldsymbol{\phi}}$ in the limit when ℓ and w both tend to zero. The bound current-density at $(r=R,\theta,\phi)$ is thus given by $\boldsymbol{J}_{\text{bound}}^{(e)} = \mu_o^{-1}(M_o\sin\theta/w)\hat{\boldsymbol{\phi}}$. Since the current is confined to a thin layer on the surface, we could use the surface-current-density $\boldsymbol{J}_{\text{s-bound}}^{(e)} = w\boldsymbol{J}_{\text{bound}}^{(e)}$ instead of the bulk current-density. Consequently, $\boldsymbol{J}_{\text{s-bound}}^{(e)} = \mu_o^{-1}M_o\sin\theta\,\hat{\boldsymbol{\phi}}$.



Problem 2)

a) $\boldsymbol{E}^{(\text{total})} = \boldsymbol{E}^{(\text{inc})} + \boldsymbol{E}^{(\text{ref})} = E_o\hat{\boldsymbol{x}}\,\{\cos[(\omega/c)z-\omega t] - \cos[(\omega/c)z+\omega t]\} = 2E_o\hat{\boldsymbol{x}}\sin(\omega z/c)\sin(\omega t)$.

$\boldsymbol{H}^{(\text{total})} = \boldsymbol{H}^{(\text{inc})} + \boldsymbol{H}^{(\text{ref})} = (E_o/Z_o)\hat{\boldsymbol{y}}\,\{\cos[(\omega/c)z-\omega t] + \cos[(\omega/c)z+\omega t]\} = 2(E_o/Z_o)\hat{\boldsymbol{y}}\cos(\omega z/c)\cos(\omega t)$.

b) The E-field vanishes where $\sin(\omega z/c)=0$, that is, $z = 0, -\lambda/2, -\lambda, -3\lambda/2, \ldots$. Here $\lambda = 2\pi c/\omega$.
The H-field vanishes where $\cos(\omega z/c)=0$, that is, $z = -\lambda/4, -3\lambda/4, -5\lambda/4, \ldots$.

c) Energy density of the E-field: $\tfrac{1}{2}\varepsilon_o|\boldsymbol{E}|^2 = 2\varepsilon_o E_o^2\sin^2(\omega z/c)\sin^2(\omega t)$.

Energy density of the H-field: $\tfrac{1}{2}\mu_o|\boldsymbol{H}|^2 = 2\varepsilon_o E_o^2\cos^2(\omega z/c)\cos^2(\omega t)$.

d) $\boldsymbol{S}(z,t) = \boldsymbol{E}^{(\text{total})} \times \boldsymbol{H}^{(\text{total})} = (E_o^2/Z_o)\,\hat{\boldsymbol{z}}\,\sin(2\omega z/c)\sin(2\omega t)$.

The z-dependence of the Poynting vector, $\sin(2\omega z/c) = \sin(4\pi z/\lambda)$, reveals that $\boldsymbol{S}(z,t)$ is zero at all integer multiples of $\lambda/4$. Therefore, where either the E-field or the H-field of the standing wave has a node, no energy flows at all. The energy only flows along z in between these adjacent nodes, which are separated by intervals of $\Delta z = \lambda/4$. The time-dependence of the Poynting vector, $\sin(2\omega t)$, shows that energy flow along z changes direction at twice the optical frequency ω. There are periodic instants when the energy is entirely in the E-field, followed by instants when the energy is entirely in the H-field. In between, the energy moves either slightly to the right or slightly to the left along z, in order to maintain the E- and H-field energy profiles found in part (c).

Problem 3)

a) At the mirror surface, we have $z = 0$ and the tangential E-field is along the x-axis. Adding the x-components of the incident and reflected E-fields, we find

$$E_x^{(\text{inc})} + E_x^{(\text{ref})} = E_o \cos\theta \exp\{i(\omega/c)[(\sin\theta)x - ct]\} - E_o \cos\theta \exp\{i(\omega/c)[(\sin\theta)x - ct]\} = 0.$$

Since the fields inside the perfectly-conducting mirror are zero, the continuity of the tangential E-field requires $E_x^{(\text{total})}$ at the front facet of the mirror to vanish. This is indeed the case for the tangential component of the E-field at $z = 0$.

b) At the front facet, we have $z = 0$ and the tangential H-field is along the y-axis. Adding the y-components of the incident and reflected H-fields, we find

$$H_y^{(\text{inc})} + H_y^{(\text{ref})} = 2(E_o/Z_o)\exp\{i(\omega/c)[(\sin\theta)x - ct]\}.$$

Since the H-field within the perfectly-conducting mirror is zero, the discontinuity of H_y must be accounted for by the presence of a surface-current-density whose magnitude is equal to H_y at the mirror surface, and whose direction, while perpendicular to the H-field, follows the right-hand rule. We will have

$$\boldsymbol{J}_s(x, y, z = 0, t) = 2(E_o/Z_o)\hat{\boldsymbol{x}}\,\exp\{i(\omega/c)[(\sin\theta)x - ct]\}.$$

c) At the front facet, we have $z = 0$ and the perpendicular E-field is along the z-axis. Adding the z-components of the incident and reflected E-fields, we find

$$E_z^{(\text{inc})} + E_z^{(\text{ref})} = -2E_o \sin\theta \exp\{i(\omega/c)[(\sin\theta)x - ct]\}.$$

Since the E-field within the perfectly-conducting mirror is zero, the discontinuity of E_z must be accounted for by the presence of a surface-charge-density whose magnitude is equal to $\varepsilon_o E_z$ at the mirror surface. We find

$$\sigma_s(x, y, z = 0, t) = 2\varepsilon_o E_o \sin\theta\, \exp\{i(\omega/c)[(\sin\theta)x - ct]\}.$$

d) Charge-current continuity equation:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{J}_s + \partial\sigma_s/\partial t = \partial J_{sx}/\partial x + \partial\sigma_s/\partial t &= 2i(\omega/c)\sin\theta\,(E_o/Z_o)\,\exp\{i(\omega/c)[(\sin\theta)x - ct]\}\\ &\quad - 2i\omega\varepsilon_o E_o \sin\theta\,\exp\{i(\omega/c)[(\sin\theta)x - ct]\}\\ &= 2i\omega(\varepsilon_o - \varepsilon_o)E_o \sin\theta\,\exp\{i(\omega/c)[(\sin\theta)x - ct]\} = 0.\end{aligned}$$
