Problem 1) a) We use the dispersion relation to find k_z in terms of k_x, ω, and material parameters for each plane-wave. We then proceed to relate the various components of the E- and H-fields to each other and to the k-vector through the use of Maxwell's equations. The dispersion relation is

$$k^2 = k_x^2 + k_y^2 + k_z^2 = (\omega/c)^2 \mu(\omega)\varepsilon(\omega) \quad \rightarrow \quad k_z = \pm\sqrt{(\omega/c)^2 \mu(\omega)\varepsilon(\omega) - k_x^2 - k_y^2}. \qquad (1)$$

Considering that $k_y = 0$, and using the relevant parameters for each of the two media, we find

$$k_z^{\mathrm{i}} = -(\omega/c)\sqrt{\mu_{\mathrm{a}}(\omega)\varepsilon_{\mathrm{a}}(\omega) - (ck_x/\omega)^2}\,; \qquad (2a)$$

$$k_z^{\mathrm{r}} = (\omega/c)\sqrt{\mu_{\mathrm{a}}(\omega)\varepsilon_{\mathrm{a}}(\omega) - (ck_x/\omega)^2}\,; \qquad (2b)$$

$$k_z^{\mathrm{t}} = -(\omega/c)\sqrt{\mu_{\mathrm{b}}(\omega)\varepsilon_{\mathrm{b}}(\omega) - (ck_x/\omega)^2}\,; \qquad (2c)$$

The choice of sign for the square root must be made such that the imaginary part of k_z is positive for upward-propagating waves, and negative for downward-propagating waves.

For p-polarized light, Maxwell's 1st equation yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_p = 0 \quad\rightarrow\quad k_x E_{xp} + k_z E_{zp} = 0 \quad\rightarrow\quad \begin{cases} E_{zp}^{\mathrm{i}} = -(k_x/k_z^{\mathrm{i}})E_{xp}^{\mathrm{i}} \\ E_{zp}^{\mathrm{r}} = -(k_x/k_z^{\mathrm{r}})E_{xp}^{\mathrm{r}} \\ E_{zp}^{\mathrm{t}} = -(k_x/k_z^{\mathrm{t}})E_{xp}^{\mathrm{t}} \end{cases} \qquad (3)$$

As for the H-field of the various p-polarized beams, we use Maxwell's 3rd equation to write

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{E} = \mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H} \quad\rightarrow\quad k_z E_{xp} - k_x E_{zp} = \mu_{\mathrm{o}}\mu(\omega)\omega H_{yp}$$

$$\rightarrow\quad k_z E_{xp} + (k_x^2/k_z)E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\omega H_{yp} \quad\rightarrow\quad (k_x^2 + k_z^2)E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\omega k_z H_{yp}$$

$$\rightarrow\quad (\omega/c)^2\mu(\omega)\varepsilon(\omega)E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\omega k_z H_{yp} \quad\rightarrow\quad (\omega/c)\varepsilon(\omega)E_{xp} = \mu_{\mathrm{o}} c k_z H_{yp}$$

$$\rightarrow\quad H_{yp} = \frac{(\omega/c)\varepsilon(\omega)}{Z_{\mathrm{o}}k_z}E_{xp} \quad\rightarrow\quad \begin{cases} H_{yp}^{\mathrm{i}} = \dfrac{(\omega/c)\varepsilon_{\mathrm{a}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{i}}}E_{xp}^{\mathrm{i}} \\ H_{yp}^{\mathrm{r}} = \dfrac{(\omega/c)\varepsilon_{\mathrm{a}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{r}}}E_{xp}^{\mathrm{r}} \\ H_{yp}^{\mathrm{t}} = \dfrac{(\omega/c)\varepsilon_{\mathrm{b}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{t}}}E_{xp}^{\mathrm{t}} \end{cases} \qquad (4)$$

For the s-polarized light, we use Maxwell's 4th equation to relate H_z to H_x, as follows:

$$\boldsymbol{k}\cdot\boldsymbol{H} = 0 \quad\rightarrow\quad k_x H_{xs} + k_z H_{zs} = 0 \quad\rightarrow\quad \begin{cases} H_{zs}^{\mathrm{i}} = -(k_x/k_z^{\mathrm{i}})H_{xs}^{\mathrm{i}} \\ H_{zs}^{\mathrm{r}} = -(k_x/k_z^{\mathrm{r}})H_{xs}^{\mathrm{r}} \\ H_{zs}^{\mathrm{t}} = -(k_x/k_z^{\mathrm{t}})H_{xs}^{\mathrm{t}} \end{cases} \qquad (5)$$

The E-field of the s-polarized beam is readily obtained from Maxwell's 2nd equation, that is,

$$\boldsymbol{\nabla}\times\boldsymbol{H}=\partial\boldsymbol{D}/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{H}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega\boldsymbol{E} \quad\rightarrow\quad k_z H_{xs}-k_x H_{zs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega E_{ys}$$

$$\rightarrow\quad k_z H_{xs}+(k_x^2/k_z)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega E_{ys} \quad\rightarrow\quad (k_x^2+k_z^2)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\,\omega k_z E_{ys}$$

$$\rightarrow\quad (\omega/c)^2\mu(\omega)\varepsilon(\omega)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega k_z E_{ys} \quad\rightarrow\quad (\omega/c)\mu(\omega)H_{xs}=-\varepsilon_{\text{o}}ck_z E_{ys}$$

$$\rightarrow\quad E_{ys}=-\frac{(\omega/c)\mu(\omega)}{k_z}Z_{\text{o}}H_{xs} \quad\rightarrow\quad \begin{cases} E_{ys}^{\text{i}}=-\dfrac{(\omega/c)\,\mu_{\text{a}}(\omega)}{k_z^{\text{i}}}Z_{\text{o}}H_{xs}^{\text{i}} \\[2ex] E_{ys}^{\text{r}}=-\dfrac{(\omega/c)\,\mu_{\text{a}}(\omega)}{k_z^{\text{r}}}Z_{\text{o}}H_{xs}^{\text{r}} \\[2ex] E_{ys}^{\text{t}}=-\dfrac{(\omega/c)\,\mu_{\text{b}}(\omega)}{k_z^{\text{t}}}Z_{\text{o}}H_{xs}^{\text{t}} \end{cases} \qquad (6)$$

b) For p-polarized light, the continuity of E_x and D_z at the $z=0$ interface yields

$$\begin{cases} E_{xp}^{\text{i}}+E_{xp}^{\text{r}}=E_{xp}^{\text{t}} \\ D_{zp}^{\text{i}}+D_{zp}^{\text{r}}=D_{zp}^{\text{t}} \end{cases} \rightarrow \begin{cases} E_{xp}^{\text{i}}+E_{xp}^{\text{r}}=E_{xp}^{\text{t}} & (7\text{a}) \\ \varepsilon_{\text{o}}\varepsilon_{\text{a}}E_{zp}^{\text{i}}+\varepsilon_{\text{o}}\varepsilon_{\text{a}}E_{zp}^{\text{r}}=\varepsilon_{\text{o}}\varepsilon_{\text{b}}E_{zp}^{\text{t}} & (7\text{b}) \end{cases}$$

Use Eq. (3) in Eq. (7b), then substitute for E_{xp}^{t} from Eq. (7a).

$$\rightarrow\quad (\varepsilon_{\text{a}}k_x/k_z^{\text{i}})E_{xp}^{\text{i}}+(\varepsilon_{\text{a}}k_x/k_z^{\text{r}})E_{xp}^{\text{r}}=(\varepsilon_{\text{b}}k_x/k_z^{\text{t}})(E_{xp}^{\text{i}}+E_{xp}^{\text{r}})$$

$$\rightarrow\quad [(\varepsilon_{\text{a}}/k_z^{\text{r}})-(\varepsilon_{\text{b}}/k_z^{\text{t}})]E_{xp}^{\text{r}}=[(\varepsilon_{\text{b}}/k_z^{\text{t}})-(\varepsilon_{\text{a}}/k_z^{\text{i}})]E_{xp}^{\text{i}}$$

Use Eqs. (2a, 2b) to set $k_z^{\text{r}}=-k_z^{\text{i}}$.

$$\rightarrow\quad \rho_p=\frac{E_{xp}^{\text{r}}}{E_{xp}^{\text{i}}}=\frac{(\varepsilon_{\text{b}}/k_z^{\text{t}})-(\varepsilon_{\text{a}}/k_z^{\text{i}})}{(\varepsilon_{\text{a}}/k_z^{\text{r}})-(\varepsilon_{\text{b}}/k_z^{\text{t}})}=\frac{\varepsilon_{\text{a}}k_z^{\text{t}}-\varepsilon_{\text{b}}k_z^{\text{i}}}{\varepsilon_{\text{a}}k_z^{\text{t}}+\varepsilon_{\text{b}}k_z^{\text{i}}}. \qquad (8)$$

The transmission coefficient τ_p is found from Eqs. (7a) and (8), as follows:

$$\tau_p=E_{xp}^{\text{t}}/E_{xp}^{\text{i}}=1+(E_{xp}^{\text{r}}/E_{xp}^{\text{i}})=1+\rho_p=\frac{2\varepsilon_{\text{a}}k_z^{\text{t}}}{\varepsilon_{\text{a}}k_z^{\text{t}}+\varepsilon_{\text{b}}k_z^{\text{i}}}. \qquad (9)$$

c) For s-polarized light, the continuity of H_x and B_z at the $z=0$ interface yields

$$\begin{cases} H_{xs}^{\text{i}}+H_{xs}^{\text{r}}=H_{xs}^{\text{t}} \\ B_{zs}^{\text{i}}+B_{zs}^{\text{r}}=B_{zs}^{\text{t}} \end{cases} \rightarrow \begin{cases} H_{xs}^{\text{i}}+H_{xs}^{\text{r}}=H_{xs}^{\text{t}} & (10\text{a}) \\ \mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{i}}+\mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{r}}=\mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{t}} & (10\text{b}) \end{cases}$$

Use Eq. (5) in Eq. (10b), then substitute for H_{xs}^{t} from Eq. (10a).

$$\rightarrow\quad (\mu_{\text{a}}k_x/k_z^{\text{i}})H_{xs}^{\text{i}}+(\mu_{\text{a}}k_x/k_z^{\text{r}})H_{xs}^{\text{r}}=(\mu_{\text{b}}k_x/k_z^{\text{t}})(H_{xs}^{\text{i}}+H_{xs}^{\text{r}})$$

$$\rightarrow\quad [(\mu_{\text{a}}/k_z^{\text{r}})-(\mu_{\text{b}}/k_z^{\text{t}})]H_{xs}^{\text{r}}=[(\mu_{\text{b}}/k_z^{\text{t}})-(\mu_{\text{a}}/k_z^{\text{i}})]H_{xs}^{\text{i}}$$

Use Eqs. (2a, 2b) to set $k_z^{\text{r}}=-k_z^{\text{i}}$.

$$\rightarrow\quad \frac{H_{xs}^{\text{r}}}{H_{xs}^{\text{i}}}=\frac{(\mu_{\text{b}}/k_z^{\text{t}})-(\mu_{\text{a}}/k_z^{\text{i}})}{(\mu_{\text{a}}/k_z^{\text{r}})-(\mu_{\text{b}}/k_z^{\text{t}})}=\frac{\mu_{\text{a}}k_z^{\text{t}}-\mu_{\text{b}}k_z^{\text{i}}}{\mu_{\text{a}}k_z^{\text{t}}+\mu_{\text{b}}k_z^{\text{i}}}. \qquad (11)$$

The Fresnel reflection coefficient for s-polarized light is defined as $\rho_s = E^{\mathrm{r}}_{ys} / E^{\mathrm{i}}_{ys}$. From Eq.(6), it is clear that $\rho_s = -H^{\mathrm{r}}_{xs} / H^{\mathrm{i}}_{xs}$. Therefore,

$$\rho_s = \frac{\mu_{\mathrm{b}} k_z^{\mathrm{i}} - \mu_{\mathrm{a}} k_z^{\mathrm{t}}}{\mu_{\mathrm{b}} k_z^{\mathrm{i}} + \mu_{\mathrm{a}} k_z^{\mathrm{t}}}. \tag{12}$$

The transmission coefficient for the H-field is found from Eqs.(10a) and (11), as follows:

$$H^{\mathrm{t}}_{xs} / H^{\mathrm{i}}_{xs} = 1 + (H^{\mathrm{r}}_{xs} / H^{\mathrm{i}}_{xs}) = \frac{2\mu_{\mathrm{a}} k_z^{\mathrm{t}}}{\mu_{\mathrm{a}} k_z^{\mathrm{t}} + \mu_{\mathrm{b}} k_z^{\mathrm{i}}}. \tag{13}$$

The Fresnel transmission coefficient for s-polarized light, being defined as $\tau_s = E^{\mathrm{t}}_{ys} / E^{\mathrm{i}}_{ys}$, may now be found from Eq.(6) as $\tau_s = (\mu_{\mathrm{b}} k_z^{\mathrm{i}} / \mu_{\mathrm{a}} k_z^{\mathrm{t}}) H^{\mathrm{t}}_{xs} / H^{\mathrm{i}}_{xs}$. Consequently

$$\tau_s = \frac{2\mu_{\mathrm{b}} k_z^{\mathrm{i}}}{\mu_{\mathrm{a}} k_z^{\mathrm{t}} + \mu_{\mathrm{b}} k_z^{\mathrm{i}}}. \tag{14}$$

Problem 2) a) Within the incidence medium, the x-component of the k-vector is given by $k_x = (\omega/c)\, n \sin\theta^{\mathrm{i}}$. The Fresnel transmission coefficient τ_s thus yields the E-field amplitude transmitted into the free-space region below the prism, as follows:

$$\tau_s = E^{\mathrm{t}}_{y\mathrm{o}} / E^{\mathrm{i}}_{y\mathrm{o}} = \frac{2\mu_{\mathrm{b}} \sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2}}{\mu_{\mathrm{b}} \sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2} + \mu_{\mathrm{a}} \sqrt{\mu_{\mathrm{b}}\varepsilon_{\mathrm{b}} - (ck_x/\omega)^2}} = \frac{2\cos\theta^{\mathrm{i}}}{\cos\theta^{\mathrm{i}} + \mathrm{i}\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}}. \tag{1}$$

Note that the z-component of the evanescent field's k-vector, a purely imaginary entity, is given by

$$k_z^{\mathrm{t}} = -\sqrt{(\omega/c)^2 - k_x^2} = -(\omega/c)\sqrt{1 - n^2 \sin^2\theta^{\mathrm{i}}} = -\mathrm{i}\,(\omega/c)\, n \sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,. \tag{2}$$

The evanescent wave's H-field may now be calculated using Maxwell's 3rd equation, namely,

$$\boldsymbol{k} \times \boldsymbol{E}_{\mathrm{o}} = \mu_{\mathrm{o}} \omega \boldsymbol{H}_{\mathrm{o}} \quad \rightarrow \quad k_x E_{y\mathrm{o}} \hat{\boldsymbol{z}} - k_z E_{y\mathrm{o}} \hat{\boldsymbol{x}} = \mu_{\mathrm{o}} \omega \boldsymbol{H}_{\mathrm{o}}$$

$$\rightarrow \quad (\omega/c)\, n \sin\theta^{\mathrm{i}} E^{\mathrm{t}}_{y\mathrm{o}} \hat{\boldsymbol{z}} + \mathrm{i}\,(\omega/c)\, n \sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\, E^{\mathrm{t}}_{y\mathrm{o}} \hat{\boldsymbol{x}} = \mu_{\mathrm{o}} \omega \boldsymbol{H}^{\mathrm{t}}_{\mathrm{o}}$$

$$\rightarrow \quad \boldsymbol{H}^{\mathrm{t}}_{\mathrm{o}} = \left[\mathrm{i}\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\, \hat{\boldsymbol{x}} + \sin\theta^{\mathrm{i}} \hat{\boldsymbol{z}}\right] n E^{\mathrm{t}}_{y\mathrm{o}} / Z_{\mathrm{o}}\,. \tag{3}$$

The complete expressions for the E- and H-fields of the evanescent wave are thus found to be

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}, t) = \mathrm{Re}\left\{\tau_s E^{\mathrm{i}}_{y\mathrm{o}} \hat{\boldsymbol{y}} \exp[\mathrm{i}(k_x x + k_z^{\mathrm{t}} z - \omega t)]\right\}, \tag{4a}$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}, t) = \mathrm{Re}\left\{\left[\mathrm{i}\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\, \hat{\boldsymbol{x}} + \sin\theta^{\mathrm{i}} \hat{\boldsymbol{z}}\right](n \tau_s E^{\mathrm{i}}_{y\mathrm{o}} / Z_{\mathrm{o}}) \exp[\mathrm{i}(k_x x + k_z^{\mathrm{t}} z - \omega t)]\right\}. \tag{4b}$$

b) Noting that $\tau_s = |\tau_s| \exp(\mathrm{i}\phi_{\tau_s})$, where

$$|\tau_s| = 2\cos\theta^{\mathrm{i}}/\cos\theta_c, \tag{5a}$$

$$\phi_{\tau_s} = -\tan^{-1}\left(\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,/\cos\theta^{\mathrm{i}}\right), \tag{5b}$$

we write the energy-density of the electromagnetic field at all points (x,y,z,t), where $z<0$, as follows:

$$\mathcal{E}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_{\mathrm{o}}|\boldsymbol{E}|^2 + \tfrac{1}{2}\mu_{\mathrm{o}}|\boldsymbol{H}|^2 = \tfrac{1}{2}|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp(2\mathrm{i}k^{\mathrm{t}}_z z)\left\{\varepsilon_{\mathrm{o}}\cos^2(k_x x - \omega t + \phi_{\tau_s})\right.$$
$$\left. + \mu_{\mathrm{o}}(n^2/Z^2_{\mathrm{o}})[(\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c)\sin^2(k_x x - \omega t + \phi_{\tau_s}) + \sin^2\theta^{\mathrm{i}}\cos^2(k_x x - \omega t + \phi_{\tau_s})]\right\}. \tag{6}$$

Substitution for k_z^{t} from Eq. (2) and setting $n\sin\theta_c = 1$ simplifies the above equation, yielding

$$\mathcal{E}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_{\mathrm{o}}|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,z\right]\{\cos[2(k_x x - \omega t + \phi_{\tau_s})] + n^2\sin^2\theta^{\mathrm{i}}\}. \tag{7}$$

Next, we calculate the Poynting vector of the evanescent field, as follows:

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) = (n/Z_{\mathrm{o}})|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp(2\mathrm{i}k^{\mathrm{t}}_z z)\left\{\cos(k_x x - \omega t + \phi_{\tau_s})\hat{\boldsymbol{y}}\right.$$
$$\left.\times\left[-\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,\sin(k_x x - \omega t + \phi_{\tau_s})\hat{\boldsymbol{x}} + \sin\theta^{\mathrm{i}}\cos(k_x x - \omega t + \phi_{\tau_s})\hat{\boldsymbol{z}}\right]\right\}. \tag{8}$$

Substitution for k_z^{t} from Eq. (2), followed by further algebraic manipulations, simplify the above equation, yielding

$$\boldsymbol{S}(\boldsymbol{r},t) = (n/Z_{\mathrm{o}})|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,z\right]$$
$$\times\left\{\sin\theta^{\mathrm{i}}\cos^2(k_x x - \omega t + \phi_{\tau_s})\hat{\boldsymbol{x}} + \tfrac{1}{2}\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,\sin[2(k_x x - \omega t + \phi_{\tau_s})]\hat{\boldsymbol{z}}\right\}. \tag{9}$$

To verify the energy continuity equation, we calculate its two terms separately, namely,

$$\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) = \frac{\partial S_x}{\partial x} + \frac{\partial S_z}{\partial z} = (n/Z_{\mathrm{o}})|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,z\right]$$
$$\times\left\{-k_x\sin\theta^{\mathrm{i}}\sin[2(k_x x - \omega t + \phi_{\tau_s})] + (\omega/c)n(\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c)\sin[2(k_x x - \omega t + \phi_{\tau_s})]\right\}. \tag{10}$$

Considering that $k_x = (\omega/c)n\sin\theta^{\mathrm{i}}$ and $n\sin\theta_c = 1$, the above equation simplifies, yielding

$$\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) = -\varepsilon_{\mathrm{o}}\omega|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,z\right]\sin[2(k_x x - \omega t + \phi_{\tau_s})]. \tag{11}$$

Next, we calculate the time-derivative of the energy-density given by Eq. (7). We find

$$\frac{\partial\mathcal{E}(\boldsymbol{r},t)}{\partial t} = \varepsilon_{\mathrm{o}}\omega|\tau_s|^2|E^{\mathrm{i}}_{y\mathrm{o}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\,z\right]\sin[2(k_x x - \omega t + \phi_{\tau_s})]. \tag{12}$$

It is now easy to verify that the continuity equation holds, that is, $\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) + \partial\mathcal{E}(\boldsymbol{r},t)/\partial t = 0$.

c) The time-averaged Poynting vector is readily obtained from Eq. (9), that is,

$$<\boldsymbol{S}(\boldsymbol{r},t)> = (n/Z_o)|\tau_s|^2|E_{yo}^{\mathrm{i}}|^2 \exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c}\, z\right]$$

$$\times\left\{\sin\theta^{\mathrm{i}} <\cos^2(k_x x-\omega t+\phi_{\tau_s})>\hat{\boldsymbol{x}} + \tfrac{1}{2}\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c} <\sin[2(k_x x-\omega t+\phi_{\tau_s})]>\hat{\boldsymbol{z}}\right\}.$$

$$= \tfrac{1}{2}(n/Z_o)|\tau_s|^2|E_{yo}^{\mathrm{i}}|^2 \sin\theta^{\mathrm{i}} \exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c}\, z\right]\hat{\boldsymbol{x}}. \qquad (13)$$

Cleary, the time-averaged z-component of the Poynting vector is zero, whereas its x-component is a positive entity.

d) The stored areal energy-density (per unit area of the xy-plane) is obtained by integrating the time-averaged volumetric energy-density, namely, $<\mathcal{E}(\boldsymbol{r},t)>$, along the z-axis, from $z=-\infty$ to $z=0$. We find

$$\int_{-\infty}^{0} <\mathcal{E}(x,y,z,t)>\mathrm{d}z = \tfrac{1}{2}\varepsilon_o|\tau_s|^2|E_{yo}^{\mathrm{i}}|^2\{<\cos[2(k_x x-\omega t+\phi_{\tau_s})]> + n^2\sin^2\theta^{\mathrm{i}}\}$$

$$\times\int_{-\infty}^{0}\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c}\, z\right]\mathrm{d}z$$

$$= \frac{\varepsilon_o n^2\sin^2\theta^{\mathrm{i}}|\tau_s|^2|E_{yo}^{\mathrm{i}}|^2}{4(\omega/c)n\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c}} = \frac{n(\sin 2\theta^{\mathrm{i}}/\cos\theta_c)^2|E_{yo}^{\mathrm{i}}|^2}{4Z_o\omega\sqrt{\sin^2\theta^{\mathrm{i}}-\sin^2\theta_c}}. \qquad (14)$$

Note that the stored energy-density increases indefinitely as θ^{i} approaches θ_c from above.

Problem 3) a) Since the units of $\boldsymbol{M}(\boldsymbol{r},t)$ are weber/m^2 and the delta-function has units of 1/m, the coefficient M_{so} must have the units of weber/m.

b) In the absence of $\rho_{\text{free}}(\boldsymbol{r},t)$ and $\boldsymbol{P}(\boldsymbol{r},t)$, the bound electric charge-density of the magnetized sheet is zero, that is, $\rho_{\text{bound}}^{(e)} = 0$, while the bound current-density is given by

$$\boldsymbol{J}_{\text{bound}}^{(e)} = \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t) = \mu_o^{-1}(\partial M_z/\partial y)\hat{\boldsymbol{x}} = \mu_o^{-1}M_{so}\delta'(y)\cos(\omega_o t)\hat{\boldsymbol{x}}.$$

c) Since the electric charge-density of the sheet is zero everywhere, we have $\psi(\boldsymbol{r},t)=0$. As for the vector potential, we use the symmetry of the problem and compute $\boldsymbol{A}(\boldsymbol{r},t)$ only at $(x=0,y,z=0)$, as follows:

$$\boldsymbol{A}(\boldsymbol{r},t) = (\mu_o/4\pi)\int_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{bound}}^{(m)}(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'$$

$$= \frac{M_{so}\hat{\boldsymbol{x}}}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta'(y')\cos\left[\omega_o\left(t-\sqrt{x'^2+(y-y')^2+z'^2}/c\right)\right]}{\sqrt{x'^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$= \frac{M_{so}\hat{\boldsymbol{x}}}{4\pi}\left\{\cos(\omega_o t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta'(y')\mathrm{d}y'\mathrm{d}x'\int_{-\infty}^{\infty}\frac{\cos\{(\omega_o/c)\sqrt{x'^2+(y-y')^2+z'^2}\}}{\sqrt{x'^2+(y-y')^2+z'^2}}\mathrm{d}z'\right.$$

$$\left.+\sin(\omega_o t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta'(y')\mathrm{d}y'\mathrm{d}x'\int_{-\infty}^{\infty}\frac{\sin\{(\omega_o/c)\sqrt{x'^2+(y-y')^2+z'^2}\}}{\sqrt{x'^2+(y-y')^2+z'^2}}\mathrm{d}z'\right\}$$

G&R 3.876-1,2 →

$$= \tfrac{1}{4}M_{so}\hat{\boldsymbol{x}}\left\{-\cos(\omega_o t)\int_{-\infty}^{\infty}\delta'(y')\mathrm{d}y'\int_{-\infty}^{\infty}Y_0\left[(\omega_o/c)\sqrt{x'^2+(y-y')^2}\right]\mathrm{d}x'\right.$$

$$\left.+\sin(\omega_o t)\int_{-\infty}^{\infty}\delta'(y')\mathrm{d}y'\int_{-\infty}^{\infty}J_0\left[(\omega_o/c)\sqrt{x'^2+(y-y')^2}\right]\mathrm{d}x'\right\}$$

G&R 6.677-3,4 →

$$= \tfrac{1}{2}M_{so}\hat{\boldsymbol{x}}\left\{-(\omega_o/c)^{-1}\cos(\omega_o t)\int_{-\infty}^{\infty}\delta'(y')\sin\left[(\omega_o/c)\sqrt{(y-y')^2}\right]\mathrm{d}y'\right.$$

$$\left.+(\omega_o/c)^{-1}\sin(\omega_o t)\int_{-\infty}^{\infty}\delta'(y')\cos\left[(\omega_o/c)\sqrt{(y-y')^2}\right]\mathrm{d}y'\right\}$$

$$= -\tfrac{1}{2}M_{so}\hat{\boldsymbol{x}}\,\mathrm{sign}(y)\left[\cos(\omega_o t)\cos(\omega_o|y|/c)+\sin(\omega_o t)\sin(\omega_o|y|/c)\right]$$ ← Sifting property of $\delta'(\cdot)$

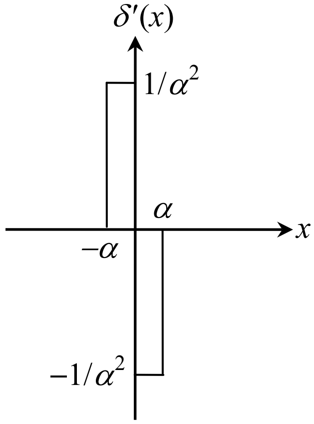
$$= -\tfrac{1}{2}M_{so}\,\mathrm{sign}(y)\cos[\omega_o(t-|y|/c)]\hat{\boldsymbol{x}}.$$

d) $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t = -\tfrac{1}{2}\,\omega_o M_{so}\,\mathrm{sign}(y)\sin[\omega_o(t-|y|/c)]\hat{\boldsymbol{x}}.$

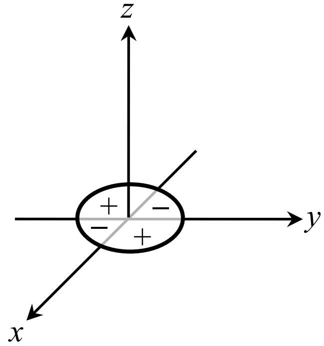
$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A} = -(\partial A_x/\partial y)\hat{\boldsymbol{z}} = \tfrac{1}{2}(\omega_o/c)M_{so}\sin[\omega_o(t-|y|/c)]\hat{\boldsymbol{z}}.$

Problem 4) a) As shown in figure (a) below, the function $\delta'(x)$ is positive when x is negative, and negative when x is positive. Therefore, the product $\delta'(x)\delta'(y)$ is positive in the first and third quadrants of the xy-plane, and negative in the second and fourth quadrants; see figure (b).

(a)



(b)



b) The charge-density ρ is in units of coulomb/m^3. Since $\delta'(x)$ and $\delta'(y)$ have units of 1/m^2, while the units of $\delta(x)$ are 1/m, we conclude that the units of Q must be coulomb$\cdot$m^2.

c) The scalar potential of the quadrupole may be calculated with the aid of the sifting property of the delta-function and its derivative. We will have

$$\psi(\boldsymbol{r}) = (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}[\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}'$$

$$= (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q\delta'(x')\delta'(y')\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \leftarrow \text{Sifting property of } \delta(z')$$

$$= \frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta'(x')\delta'(y')}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\mathrm{d}x'\mathrm{d}y' \leftarrow \text{Sifting property of } \delta'(y')$$

$$= -\frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{y\delta'(x')}{[(x-x')^2+y^2+z^2]^{3/2}}\mathrm{d}x' \leftarrow \text{Sifting property of } \delta'(x')$$

$$= \frac{3Qxy}{4\pi\varepsilon_o(x^2+y^2+z^2)^{5/2}} = \frac{3Q\sin^2\theta\sin\phi\cos\phi}{4\pi\varepsilon_o r^3} = \frac{3Q\sin^2\theta\sin 2\phi}{8\pi\varepsilon_o r^3}. \leftarrow$$

Spherical coordinates:
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Note that the potential drops with the cube of the distance r from the origin, in contrast with a point-charge, whose potential drops as $1/r$, or a point-dipole, whose potential drops as $1/r^2$.

d) $$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(r,\theta,\phi) = -\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} - \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} - \frac{\partial\psi}{r\sin\theta\partial\phi}\hat{\boldsymbol{\phi}}$$

$$= \frac{9Q\sin^2\theta\sin 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{r}} - \frac{6Q\sin\theta\cos\theta\sin 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{\theta}} - \frac{6Q\sin\theta\cos 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{\phi}}$$

$$= \frac{3Q\sin\theta}{8\pi\varepsilon_o r^4}[3\sin\theta\sin(2\phi)\hat{\boldsymbol{r}} - 2\cos\theta\sin(2\phi)\hat{\boldsymbol{\theta}} - 2\cos(2\phi)\hat{\boldsymbol{\phi}}].$$
