Opti 501 Final Exam Solutions Fall 2011

Problem 1) a) In the limit when $\alpha\to\infty$, the symmetric function $g_\alpha(x)$ becomes tall and narrow, with an area that is always equal to 2. Therefore, $\lim_{\alpha\to\infty} g_\alpha(x)=2\delta(x)$.

Odd terms omitted, as they integrate to zero.

$$\text{b) } \int_{-\infty}^{\infty} f(x)g_\alpha(x)\mathrm{d}x=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} f_n\frac{x^n}{n!}\alpha\exp(-\alpha|x|)\mathrm{d}x=2\alpha\int_0^{\infty}\sum_{n=0}^{\infty} f_{2n}\frac{x^{2n}}{(2n)!}\exp(-\alpha x)\mathrm{d}x$$

$$=2\alpha\sum_{n=0}^{\infty}\frac{f_{2n}}{(2n)!}\int_0^{\infty}x^{2n}\exp(-\alpha x)\mathrm{d}x=2\alpha\sum_{n=0}^{\infty}\frac{f_{2n}}{\alpha^{2n+1}}=2f_0+2\sum_{n=1}^{\infty}\frac{f_{2n}}{\alpha^{2n}}.$$

In the limit $\alpha\to\infty$, all the terms under the summation sign vanish, leaving $2f_0=2f(x=0)$ as the value of the integral. This, of course, is just a manifestation of the sifting property of $2\delta(x)$.

c) The function $h_\beta(x)$ is the derivative of $-½\sqrt{\beta}\exp(-\beta x^2)$, which approaches $-½\sqrt{\pi}\,\delta(x)$ in the limit when $\beta\to\infty$. Therefore, the limit of $h_\beta(x)$ is the delta-function-derivative $-½\sqrt{\pi}\,\delta'(x)$.

Even terms omitted, as they integrate to zero.

$$\text{d) } \int_{-\infty}^{\infty} f(x)h_\beta(x)\mathrm{d}x=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} f_n\frac{x^n}{n!}\beta^{3/2}x\exp(-\beta x^2)\mathrm{d}x=2\beta^{3/2}\int_0^{\infty}\sum_{n=0}^{\infty} f_{2n+1}\frac{x^{2n+1}}{(2n+1)!}x\exp(-\beta x^2)\mathrm{d}x$$

$$=2\beta^{3/2}\sum_{n=0}^{\infty}\frac{f_{2n+1}}{(2n+1)!}\int_0^{\infty}x^{2n+2}\exp(-\beta x^2)\mathrm{d}x=2\beta^{3/2}\sum_{n=0}^{\infty}\frac{f_{2n+1}}{(2n+1)!}\,\frac{(2n+1)!!\sqrt{\pi}}{2^{n+2}\beta^{n+\frac{3}{2}}}$$

$$=\tfrac{1}{2}\sqrt{\pi}\,f_1+\tfrac{1}{2}\sqrt{\pi}\sum_{n=1}^{\infty}\frac{f_{2n+1}}{n!\,(4\beta)^n}. \qquad \leftarrow \boxed{\frac{(2n+1)!!}{(2n+1)!}=\frac{1}{(2n)!!}=\frac{1}{2^n n!}}$$

When $\beta\to\infty$, the terms under the summation sign vanish, leaving $½\sqrt{\pi}f_1=½\sqrt{\pi}\,\mathrm{d}f(x)/\mathrm{d}x|_{x=0}$ as the value of the integral. This is nothing more nor less than the sifting property of $-½\sqrt{\pi}\,\delta'(x)$.

Problem 2) a) The total charge Q and the magnetic dipole moment m_z are readily found upon integration, as follows:

$$Q=4\pi R^2\sigma_{so}.$$

$$m_z=\mu_o\int_0^{\pi}\underbrace{\pi(R\sin\theta)^2}_{\text{loop area}}\underbrace{(\sigma_{so}\omega_o R\sin\theta)R\mathrm{d}\theta}_{\text{loop current}}=\pi R^4\mu_o\sigma_{so}\omega_o\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta=(\tfrac{4}{3}\pi R^3)\mu_o\sigma_{so}R\omega_o.$$

b) The E-field energy of the spherical shell is obtained by integration over the field intensity outside the shell, as the field inside is zero.

$$\mathcal{E}_E=\int_{r=R}^{\infty}\tfrac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 4\pi r^2\mathrm{d}r=\frac{Q^2}{8\pi\,\varepsilon_o R}.$$

The H-field energy has contributions from the magnetic field inside as well as that outside the shell, namely,

$$\mathcal{E}_H=\tfrac{1}{2}\mu_o\left|\frac{2m_z}{3\mu_o(4\pi R^3/3)}\hat{\boldsymbol{z}}\right|^2(4\pi R^3/3)+\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\tfrac{1}{2}\mu_o\left|\frac{m_z(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})}{4\mu_o\pi r^3}\right|^2 2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta$$

$$= \frac{m_z^2}{6\pi\mu_o R^3} + \frac{m_z^2}{16\pi\mu_o}\int_{r=R}^{\infty} r^{-4}\mathrm{d}r \int_{\theta=0}^{\pi}(4\sin\theta - 3\sin^3\theta)\,\mathrm{d}\theta = \frac{m_z^2}{6\pi\mu_o R^3} + \frac{m_z^2}{12\pi\mu_o R^3} = \frac{m_z^2}{4\pi\mu_o R^3}.$$

The H-field energy is seen to be divided between inside and outside the sphere, with the inside field containing twice as much energy as the outside field.

Inside the sphere, the Poynting vector is zero because the E-field is zero, but outside it is given by

$$\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}} \times \frac{m_z(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{4\pi\mu_o r^3} = \frac{Q m_z c^2 \sin\theta\,\hat{\boldsymbol{\phi}}}{16\pi^2 r^5}; \qquad r > R.$$

The EM angular momentum density with respect to the origin is $\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{r} \times \boldsymbol{S}(\boldsymbol{r})/c^2$; therefore, the total angular momentum of the spinning sphere may be obtained as follows:

$$\mathcal{L} = \iiint_{\text{all space}} \boldsymbol{L}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \iiint_{\substack{\text{outside}\\\text{sphere}}} \boldsymbol{r} \times \frac{Q m_z \sin\theta\,\hat{\boldsymbol{\phi}}}{16\pi^2 r^5}\mathrm{d}\boldsymbol{r} = \frac{Q m_z \hat{\boldsymbol{z}}}{8\pi}\int_{r=R}^{\infty} r^{-2}\mathrm{d}r \int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{Q m_z}{6\pi R}\hat{\boldsymbol{z}}.$$

Note that this angular momentum is purely due to the electromagnetic field; as such, it is independent of the mass of the sphere.

Problem 3)

a) $$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)], \qquad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)].$$

Incident beam: $k_x = (\omega/c)\sin\theta, \qquad k_y = 0, \qquad k_z^{\mathrm{i}} = -(\omega/c)\cos\theta.$

$$E_{xo}^{\mathrm{i}} = E_{zo}^{\mathrm{i}} = 0, \qquad E_{yo}^{\mathrm{i}} = \text{arbitrary}.$$

$$\boldsymbol{k}^{\mathrm{i}} \times \boldsymbol{E}^{\mathrm{i}} = \mu_o\mu(\omega)\omega\boldsymbol{H}^{\mathrm{i}} \rightarrow \quad (k_x\hat{\boldsymbol{x}} + k_z^{\mathrm{i}}\hat{\boldsymbol{z}}) \times E_{yo}^{\mathrm{i}}\hat{\boldsymbol{y}} = \mu_o\omega(H_{xo}^{\mathrm{i}}\hat{\boldsymbol{x}} + H_{zo}^{\mathrm{i}}\hat{\boldsymbol{z}})$$

$$\rightarrow H_{xo}^{\mathrm{i}} = \frac{\cos\theta E_{yo}^{\mathrm{i}}}{Z_o}, \quad H_{zo}^{\mathrm{i}} = \frac{\sin\theta E_{yo}^{\mathrm{i}}}{Z_o}.$$

Reflected beam: $k_x = (\omega/c)\sin\theta, \qquad k_y = 0, \qquad k_z^{\mathrm{r}} = (\omega/c)\cos\theta.$

$$E_{xo}^{\mathrm{r}} = E_{zo}^{\mathrm{r}} = 0, \qquad E_{yo}^{\mathrm{r}} = \text{unknown}.$$

$$\boldsymbol{k}^{\mathrm{r}} \times \boldsymbol{E}^{\mathrm{r}} = \mu_o\mu(\omega)\omega\boldsymbol{H}^{\mathrm{r}} \rightarrow \quad H_{xo}^{\mathrm{r}} = -\frac{\cos\theta E_{yo}^{\mathrm{r}}}{Z_o}, \quad H_{zo}^{\mathrm{r}} = \frac{\sin\theta E_{yo}^{\mathrm{r}}}{Z_o}.$$

Transmitted beam: $k_x = (\omega/c)\sin\theta, \qquad k_y = 0, \qquad k_z^{\mathrm{t}} = -\mathrm{i}(\omega/c)\sqrt{n(\omega)^2 + \sin^2\theta}\,.$

$$E_{xo}^{\mathrm{t}} = 0, \qquad \boldsymbol{k}^{\mathrm{t}}\cdot\boldsymbol{E}^{\mathrm{t}} = 0 \quad\rightarrow\quad E_{zo}^{\mathrm{t}} = -k_x E_{xo}^{\mathrm{t}}/k_z^{\mathrm{t}} = 0; \qquad E_{yo}^{\mathrm{t}} = \text{unknown}.$$

$$\boldsymbol{k}^{\mathrm{t}} \times \boldsymbol{E}^{\mathrm{t}} = \mu_o\mu(\omega)\omega\boldsymbol{H}^{\mathrm{t}} \rightarrow \quad H_{xo}^{\mathrm{t}} = \frac{\mathrm{i}\sqrt{n(\omega)^2 + \sin^2\theta}}{Z_o}E_{yo}^{\mathrm{t}}, \quad H_{zo}^{\mathrm{t}} = \frac{\sin\theta\,E_{yo}^{\mathrm{t}}}{Z_o}.$$

b) Matching the boundary conditions yields E_{yo}^{r} and E_{yo}^{t}, as follows:

$$\begin{cases} E_{yo}^{\mathrm{i}}+E_{yo}^{\mathrm{r}}=E_{yo}^{\mathrm{t}} \\ H_{xo}^{\mathrm{i}}+H_{xo}^{\mathrm{r}}=H_{xo}^{\mathrm{t}} \end{cases} \rightarrow \quad \frac{\cos\theta E_{yo}^{\mathrm{i}}}{Z_{\mathrm{o}}}-\frac{\cos\theta E_{yo}^{\mathrm{r}}}{Z_{\mathrm{o}}}=\frac{\mathrm{i}\sqrt{n^2+\sin^2\theta}}{Z_{\mathrm{o}}}(E_{yo}^{\mathrm{i}}+E_{yo}^{\mathrm{r}})$$

$$\rightarrow \quad \frac{E_{yo}^{\mathrm{r}}}{E_{yo}^{\mathrm{i}}}=\frac{\cos\theta-\mathrm{i}\sqrt{n^2+\sin^2\theta}}{\cos\theta+\mathrm{i}\sqrt{n^2+\sin^2\theta}}, \qquad \frac{E_{yo}^{\mathrm{t}}}{E_{yo}^{\mathrm{i}}}=\frac{2\cos\theta}{\cos\theta+\mathrm{i}\sqrt{n^2+\sin^2\theta}}.$$

c) The reflectance R is the absolute value of the Fresnel reflection coefficient squared, that is,

$$R=\left|\frac{E_{yo}^{\mathrm{r}}}{E_{yo}^{\mathrm{i}}}\right|^2=\frac{\cos^2\theta+(n^2+\sin^2\theta)}{\cos^2\theta+(n^2+\sin^2\theta)}=1.$$

d) The field amplitudes decay with distance z inside the plasma-like medium as $\exp[-\mathrm{Im}(k_z^{\mathrm{t}})z]$. The penetration-depth is thus a few times the inverse of $\mathrm{Im}(k_z^{\mathrm{t}})$, which is on the order of $\lambda_{\mathrm{o}}/\sqrt{n(\omega)^2+\sin^2\theta}$, where $\lambda_{\mathrm{o}}=2\pi c/\omega$ is the vacuum wavelength. Note that, if n is small, the penetration-depth at normal incidence could be large, but with an increasing θ, the penetration-depth shrinks rapidly. The E- and H-fields, of course, carry energy, having a total energy-density $\tfrac{1}{2}\varepsilon_0|\boldsymbol{E}^{\mathrm{t}}|^2+\tfrac{1}{2}\mu_0|\boldsymbol{H}^{\mathrm{t}}|^2$ inside the plasma-like medium. This energy-density must be integrated over the entire penetration-depth of the fields to yield the total energy content of the evanescent field. As for the time-averaged Poynting vector, we have

$$\begin{aligned}
<\boldsymbol{S}^{\mathrm{t}}(\boldsymbol{r},t)> &= \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\mathrm{t}*}(\boldsymbol{r},t)] \\
&= \tfrac{1}{2}\mathrm{Re}\,\{E_{yo}^{\mathrm{t}}\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{\mathrm{t}}\cdot\boldsymbol{r}-\omega t)]\times(H_{xo}^{\mathrm{t}*}\hat{\boldsymbol{x}}+H_{zo}^{\mathrm{t}*}\hat{\boldsymbol{z}})\exp[-\mathrm{i}(\boldsymbol{k}^{\mathrm{t}*}\cdot\boldsymbol{r}-\omega t)]\} \\
&= \tfrac{1}{2}\exp[-2\,\mathrm{Im}(k_z^{\mathrm{t}})z]\,\mathrm{Re}\big(E_{yo}^{\mathrm{t}}H_{zo}^{\mathrm{t}*}\hat{\boldsymbol{x}}-E_{yo}^{\mathrm{t}}H_{xo}^{\mathrm{t}*}\hat{\boldsymbol{z}}\big) \\
&= \frac{\exp\Big(2(\omega/c)\sqrt{n(\omega)^2+\sin^2\theta}\;z\Big)}{2Z_{\mathrm{o}}}\mathrm{Re}\Big(\sin\theta E_{yo}^{\mathrm{t}}E_{yo}^{\mathrm{t}*}\hat{\boldsymbol{x}}+\mathrm{i}\sqrt{n(\omega)^2+\sin^2\theta}\,E_{yo}^{\mathrm{t}}E_{yo}^{\mathrm{t}*}\hat{\boldsymbol{z}}\Big) \\
&= \frac{\exp\Big(2(\omega/c)\sqrt{n(\omega)^2+\sin^2\theta}\;z\Big)}{2Z_{\mathrm{o}}}\sin\theta\left|E_{yo}^{\mathrm{t}}\right|^2\hat{\boldsymbol{x}}.
\end{aligned}$$

Clearly, the component of the Poynting vector along the z-axis is zero, indicating that, in the steady-state, no electromagnetic energy is absorbed within the plasma-like medium. This is consistent with the 100% reflectivity obtained in part (c).

Problem 4)

a) $$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\mathrm{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)], \qquad \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\mathrm{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

Incident beam: $\quad k_x=(\omega/c)n\sin\theta, \qquad k_y=0, \qquad k_z^{\mathrm{i}}=-(\omega/c)n\cos\theta.$

$$E_{xo}^{\mathrm{i}}=\text{arbitrary}, \qquad E_{yo}^{\mathrm{i}}=0; \qquad \boldsymbol{k}^{\mathrm{i}}\cdot\boldsymbol{E}^{\mathrm{i}}=0 \;\rightarrow\; E_{zo}^{\mathrm{i}}=-k_xE_{xo}^{\mathrm{i}}/k_z^{\mathrm{i}}.$$

$$\boldsymbol{k}^{\mathrm{i}}\times\boldsymbol{E}^{\mathrm{i}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{i}}\rightarrow\quad(k_x\hat{\boldsymbol{x}}+k_z^{\mathrm{i}}\hat{\boldsymbol{z}})\times(E_{x\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{x}}+E_{z\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{z}})=\mu_{\mathrm{o}}\omega H_{y\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{y}}\rightarrow\quad k_z^{\mathrm{i}}E_{x\mathrm{o}}^{\mathrm{i}}-k_xE_{z\mathrm{o}}^{\mathrm{i}}=\mu_{\mathrm{o}}\omega H_{y\mathrm{o}}^{\mathrm{i}}$$

$$\rightarrow H_{y\mathrm{o}}^{\mathrm{i}}=\frac{k_x^2+k_z^{\mathrm{i}\,2}}{\mu_{\mathrm{o}}\omega k_z^{\mathrm{i}}}E_{x\mathrm{o}}^{\mathrm{i}}=-\frac{(\omega/c)^2n^2}{\mu_{\mathrm{o}}\omega(\omega/c)n\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}=-\frac{n}{Z_{\mathrm{o}}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}.$$

Reflected beam: $\quad k_x=(\omega/c)n\sin\theta,\qquad k_y=0,\qquad k_z^{\mathrm{r}}=(\omega/c)n\cos\theta.$

$$E_{x\mathrm{o}}^{\mathrm{r}}=\text{unknown},\qquad E_{y\mathrm{o}}^{\mathrm{r}}=0;\qquad \boldsymbol{k}^{\mathrm{r}}\cdot\boldsymbol{E}^{\mathrm{r}}=0\quad\rightarrow\quad E_{z\mathrm{o}}^{\mathrm{r}}=-k_xE_{x\mathrm{o}}^{\mathrm{r}}/k_z^{\mathrm{r}}.$$

$$\boldsymbol{k}^{\mathrm{r}}\times\boldsymbol{E}^{\mathrm{r}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{r}}\rightarrow\quad H_{y\mathrm{o}}^{\mathrm{r}}=\frac{k_x^2+k_z^{\mathrm{r}\,2}}{\mu_{\mathrm{o}}\omega k_z^{\mathrm{r}}}E_{x\mathrm{o}}^{\mathrm{r}}=\frac{n}{Z_{\mathrm{o}}\cos\theta}E_{x\mathrm{o}}^{\mathrm{r}}.$$

Evanescent beam: $\quad k_x=(\omega/c)\,n\sin\theta,\qquad k_y=0,\qquad k_z^{\mathrm{t}}=-\mathrm{i}\,(\omega/c)\sqrt{n^2\sin^2\theta-1}.$

$$E_{x\mathrm{o}}^{\mathrm{t}}=\text{unknown},\qquad E_{y\mathrm{o}}^{\mathrm{t}}=0;\qquad \boldsymbol{k}^{\mathrm{t}}\cdot\boldsymbol{E}^{\mathrm{t}}=0\quad\rightarrow\quad E_{z\mathrm{o}}^{\mathrm{t}}=-k_xE_{x\mathrm{o}}^{\mathrm{t}}/k_z^{\mathrm{t}}.$$

$$\boldsymbol{k}^{\mathrm{t}}\times\boldsymbol{E}^{\mathrm{t}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{t}}\rightarrow\quad H_{y\mathrm{o}}^{\mathrm{t}}=\frac{k_x^2+k_z^{\mathrm{t}\,2}}{\mu_{\mathrm{o}}\omega k_z^{\mathrm{t}}}E_{x\mathrm{o}}^{\mathrm{t}}=\frac{\mathrm{i}}{Z_{\mathrm{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\mathrm{o}}^{\mathrm{t}}.$$

b) Matching the boundary conditions yields the evanescent field amplitudes as follows:

$$\begin{cases}E_{x\mathrm{o}}^{\mathrm{i}}+E_{x\mathrm{o}}^{\mathrm{r}}=E_{x\mathrm{o}}^{\mathrm{t}}\\ H_{y\mathrm{o}}^{\mathrm{i}}+H_{y\mathrm{o}}^{\mathrm{r}}=H_{y\mathrm{o}}^{\mathrm{t}}\end{cases}\rightarrow\quad\begin{cases}E_{x\mathrm{o}}^{\mathrm{r}}=E_{x\mathrm{o}}^{\mathrm{t}}-E_{x\mathrm{o}}^{\mathrm{i}}\\ -\dfrac{n}{Z_{\mathrm{o}}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}+\dfrac{n}{Z_{\mathrm{o}}\cos\theta}E_{x\mathrm{o}}^{\mathrm{r}}=\dfrac{\mathrm{i}}{Z_{\mathrm{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\mathrm{o}}^{\mathrm{t}}\end{cases}$$

$$\rightarrow\quad E_{x\mathrm{o}}^{\mathrm{t}}=\frac{2n\sqrt{n^2\sin^2\theta-1}}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}$$

$$E_{z\mathrm{o}}^{\mathrm{t}}=-k_xE_{x\mathrm{o}}^{\mathrm{t}}/k_z^{\mathrm{t}}\quad\rightarrow\quad E_{z\mathrm{o}}^{\mathrm{t}}=\frac{-2\mathrm{i}\,n^2\sin\theta}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}$$

$$H_{y\mathrm{o}}^{\mathrm{t}}=\frac{\mathrm{i}}{Z_{\mathrm{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\mathrm{o}}^{\mathrm{t}}\quad\rightarrow\quad H_{y\mathrm{o}}^{\mathrm{t}}=\frac{2\mathrm{i}n\sqrt{n^2\sin^2\theta-1}}{Z_{\mathrm{o}}[n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta]}E_{x\mathrm{o}}^{\mathrm{i}}.$$

c) The time-averaged energy-density of the evanescent E-field is calculated as follows:

$$\mathcal{E}_E(\boldsymbol{r})=\tfrac{1}{2}\varepsilon_{\mathrm{o}}<\left|\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)]\right|^2>=\tfrac{1}{4}\varepsilon_{\mathrm{o}}\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r})\cdot\boldsymbol{E}^{\mathrm{t}*}(\boldsymbol{r})]=\tfrac{1}{4}\varepsilon_{\mathrm{o}}\left(|E_{x\mathrm{o}}^{\mathrm{t}}|^2+|E_{z\mathrm{o}}^{\mathrm{t}}|^2\right)\exp[-2\mathrm{Im}(k_z^{\mathrm{t}})z]$$

$$=\tfrac{1}{4}\varepsilon_{\mathrm{o}}\left(\left|\frac{2n\sqrt{n^2\sin^2\theta-1}}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}\right|^2+\left|\frac{-2\mathrm{i}\,n^2\sin\theta}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\mathrm{o}}^{\mathrm{i}}\right|^2\right)\exp\left(2(\omega/c)\sqrt{n^2\sin^2\theta-1}\,z\right)$$

$$\rightarrow\quad\mathcal{E}_E(\boldsymbol{r})=\frac{n^2(2n^2\sin^2\theta-1)}{(n^2-1)[(n^2+1)\sin^2\theta-1]}\varepsilon_{\mathrm{o}}\left|E_{x\mathrm{o}}^{\mathrm{i}}\right|^2\exp\left(2(\omega/c)\sqrt{n^2\sin^2\theta-1}\,z\right).$$

Integration over z from $-\infty$ to 0 then yields the total E-field energy per unit area of the interface, as follows:

$$\int_{-\infty}^{0} \mathcal{E}_E(r)\,\mathrm{d}z = \frac{n^2(2n^2\sin^2\theta - 1)}{2(\omega/c)(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}}\,\varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2.$$

Similarly, the time-averaged energy-density of the evanescent H-field is calculated as follows:

$$\mathcal{E}_H(\boldsymbol{r}) = \tfrac{1}{2}\mu_o < \left|\mathrm{Re}[\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t)]\right|^2 > = \tfrac{1}{4}\mu_o\,\mathrm{Re}[\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r})\cdot\boldsymbol{H}^{\mathrm{t}*}(\boldsymbol{r})] = \tfrac{1}{4}\mu_o |H_{yo}^{\mathrm{t}}|^2 \exp[-2\,\mathrm{Im}(k_z^{\mathrm{t}})z]$$

$$= \tfrac{1}{4}\mu_o \left| \frac{2\mathrm{i}n\sqrt{n^2\sin^2\theta - 1}}{Z_o[n\sqrt{n^2\sin^2\theta - 1} - \mathrm{i}\cos\theta]} E_{xo}^{\mathrm{i}} \right|^2 \exp\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\,z\right)$$

$$= \frac{n^2(n^2\sin^2\theta - 1)}{(n^2-1)[(n^2+1)\sin^2\theta - 1]}\,\varepsilon_o \left|E_{xo}^{\mathrm{i}}\right|^2 \exp\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\,z\right).$$

Integration over z from $-\infty$ to 0 then yields the total H-field energy per unit area of the interface, as follows:

$$\int_{-\infty}^{0} \mathcal{E}_H(r)\,\mathrm{d}z = \frac{n^2(n^2\sin^2\theta - 1)}{2(\omega/c)(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}}\,\varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2.$$

The total electromagnetic energy stored in the evanescent field (per unit area of the interface) may thus be obtained by adding the preceding expressions for the E- and H-field energies. Note that the energy content of the evanescent field is *not* equally split between the E- and H-fields.

If the final result is to be expressed in terms of the incident optical power P^{i} (rather than the intensity of the incident beam's E_x-component), the relation between P^{i} and $|E_{xo}^{\mathrm{i}}|^2$ is found to be

$$P^{\mathrm{i}}\hat{\boldsymbol{k}}^{\mathrm{i}} = <\boldsymbol{S}^{\mathrm{i}}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}_o^{\mathrm{i}} \times \boldsymbol{H}_o^{\mathrm{i}*}) = \tfrac{1}{2}\mathrm{Re}[(E_{xo}^{\mathrm{i}}\hat{\boldsymbol{x}} + E_{zo}^{\mathrm{i}}\hat{\boldsymbol{z}}) \times H_{yo}^{\mathrm{i}*}\hat{\boldsymbol{y}}]$$

$$= \frac{n\left|E_{xo}^{\mathrm{i}}\right|^2(\sin\theta\hat{\boldsymbol{x}} - \cos\theta\hat{\boldsymbol{z}})}{2Z_o\cos^2\theta} = \frac{n\left|E_{xo}^{\mathrm{i}}\right|^2}{2Z_o\cos^2\theta}\hat{\boldsymbol{k}}^{\mathrm{i}}.$$

Considering that the incident beam's footprint is larger than its cross-sectional area by a factor of $\cos\theta$, we may finally relate the evanescent stored energy to the incident optical power (per unit cross-sectional area of the incident beam) as follows:

$$\text{Evanescent energy} = \frac{n(3n^2\sin^2\theta - 2)\cos\theta\, P^{\mathrm{i}}}{\omega(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}}.$$
