Solution Using the method of Fourier Transform:

$z_1''(t) + \gamma z_1'(t) + \omega_0^2 z_1(t) = (1/m)\, f(t)$ ← Fourier transform both sides

$z_1(t) = \int_{-\infty}^{\infty} Z_1(s)\, e^{+i2\pi st} ds \Rightarrow z_1'(t) = \int_{-\infty}^{\infty} i2\pi s Z_1(s)\, e^{+i2\pi st} ds \Rightarrow \mathcal{F}\{z_1''(t)\} = -(2\pi s)^2 Z_1(s)$

$\Rightarrow -(2\pi s)^2 Z_1(s) + i2\pi s\gamma Z_1(s) + \omega_0^2 Z_1(s) = (1/m) F(s) \Longrightarrow$

$$Z_1(s) = \frac{(1/m) F(s)}{-(2\pi s)^2 + \omega_0^2 + i2\pi\gamma s}$$ ← Inverse F.T. to find $z_1(t)$.

Example: $f(t) = F_0\, \text{step}(t)\, e^{i2\pi ft}$ ← F_0 is complex constant; f is real-valued freq.

Of course, the actual excitation is $\text{Re}\{f(t)\}$ or $\text{Im}\{f(t)\}$, but, the solution to the actual excitation is simply the Re or Im part of the solution to $f(t)$.

$F(s) = \mathcal{F}\{f(t)\} = F_0\, \mathcal{F}\{\text{step}(t)\} * \mathcal{F}\{e^{i2\pi ft}\} = F_0\{\frac{1}{2}\delta(s) - \frac{i}{2\pi s}\} * \delta(s-f)$.

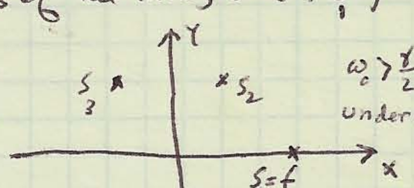
$= \frac{1}{2} F_0\, \delta(s-f) - \frac{iF_0}{2\pi(s-f)}$

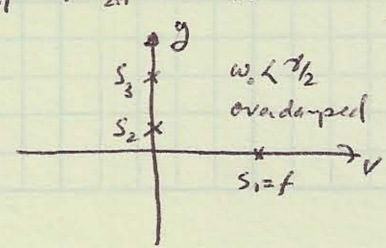
Therefore, $Z_1(s) = \frac{(F_0/2m)\,\delta(s-f)}{-4\pi^2 s^2 + \omega_0^2 + i2\pi\gamma s} - \left(\frac{iF_0}{2\pi m}\right) \frac{1}{-4\pi^2 [s^2 - i\frac{\gamma}{2\pi}s - (\frac{\omega_0}{2\pi})^2](s-f)} \Longrightarrow$

$$z_1(t) = \int_{-\infty}^{\infty} Z_1(s) e^{+i2\pi st} ds = \frac{(F_0/2m)\, e^{i2\pi ft}}{-4\pi^2 f^2 + i2\pi\gamma f + \omega_0^2} + \frac{iF_0}{(2\pi)^3 m} \int_{-\infty}^{\infty} \frac{e^{i2\pi st}\, ds}{(s-f)[s^2 - i\frac{\gamma}{2\pi}s - (\frac{\omega_0}{2\pi})^2]}$$

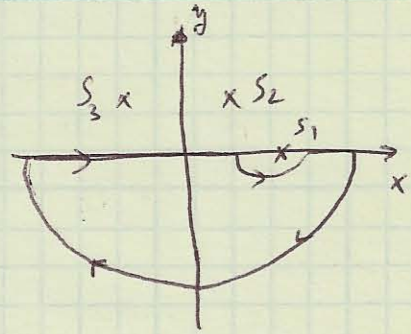
The above integral must be evaluated over the upper semi-circle when $t>0$, and over the lower semi-circle when $t<0$ (Jordan's lemma).

Poles of the integrand: $s_1 = f$ and $s_{2,3} = i\frac{\gamma}{4\pi} \pm \sqrt{(\frac{\omega_0}{2\pi})^2 - (\gamma/4\pi)^2}$

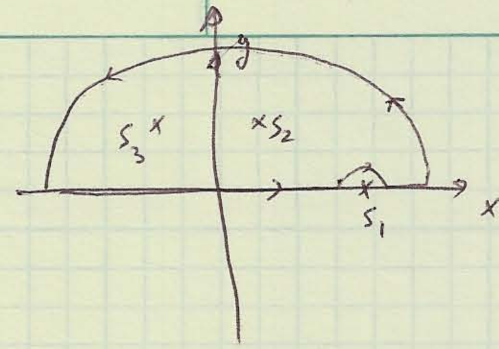




← s_2 and s_3 overlap in the case of critical damping $\omega_0 = \gamma/2$



$t<0$



(Counterclockwise)
⇓
minus

$t>0$

For $t<0$, the integral amounts to minus half the residue at $s_1=f$.

$\frac{1}{2}$ Residue at $s_1=f$: $\frac{1}{2}(2\pi i)\frac{iF_0}{(2\pi)^3 m}\frac{e^{i2\pi ft}}{f^2 - i\frac{\gamma}{2\pi}f-(\frac{\omega_0}{2\pi})^2} = \frac{-(F_0/2m)\,e^{i2\pi ft}}{4\pi^2[f^2 - i\frac{\gamma}{2\pi}f-(\frac{\omega_0}{2\pi})^2]}$

Subtracting the above from the first term in the expression for $\xi_1(t)$ yields <u>zero</u> for the function $\xi_1(t)$ when $t<0$.

For $t>0$, the integral equals <u>half</u> the residue at $s_1=f$ plus the two residues at $s=s_2$, $s=s_3$.

$$2\pi i \sum_{s_1,s_2,s_3} \text{Residues} = 2\pi i \left\{ \frac{1}{2}\frac{e^{i2\pi ft}}{f^2 - i\frac{\gamma}{2\pi}f-(\frac{\omega_0}{2\pi})^2} + \frac{e^{i2\pi s_2 t}}{(s_2-f)(s_2-s_3)} + \frac{e^{i2\pi s_3 t}}{(s_3-f)(s_3-s_2)} \right\}$$

We write the solution for the underdamped case of $\omega_0>\gamma/2$, but the other cases are just as straightforward.

$$\xi_1(t) = \frac{(F_0/2m)\,e^{i2\pi ft}}{-4\pi^2f^2+i2\pi\gamma f+\omega_0^2} - \frac{(F_0/2m)\,e^{i2\pi ft}}{4\pi^2[f^2 - i\frac{\gamma}{2\pi}f-(\frac{\omega_0}{2\pi})^2]} + \frac{(F_0/m)}{4\pi^2(s_2-s_3)}\left\{\frac{e^{i2\pi s_2 t}}{s_2-f} - \frac{e^{i2\pi s_3 t}}{s_3-f}\right\}$$

$$\Rightarrow \xi_1(t) = -\frac{(F_0/m)\,e^{i2\pi ft}}{(4\pi^2)[f^2 - i\frac{\gamma}{2\pi}f-(\frac{\omega_0}{2\pi})^2]} - \frac{(F_0/m)}{2\sqrt{\omega_0^2-\gamma^2/4}}\left\{\frac{e^{-\gamma t/2}\,e^{i\sqrt{\omega_0^2-\gamma^2/4}\,t}}{i\frac{\gamma}{2}+\sqrt{\omega_0^2-\gamma^2/4}-\underset{2\pi}{\hat{f}}} - \frac{e^{-\gamma t/2}\,e^{-i\sqrt{\omega_0^2-\gamma^2/4}\,t}}{i\frac{\gamma}{2}-\sqrt{\omega_0^2-\gamma^2/4}-\underset{2\pi}{\hat{f}}}\right\};\ t>0$$

This general solution can be checked for the initial conditions, $\xi_1(0^+)=0$ and $\xi_1'(0^+)=0$. It satisfies both these conditions.