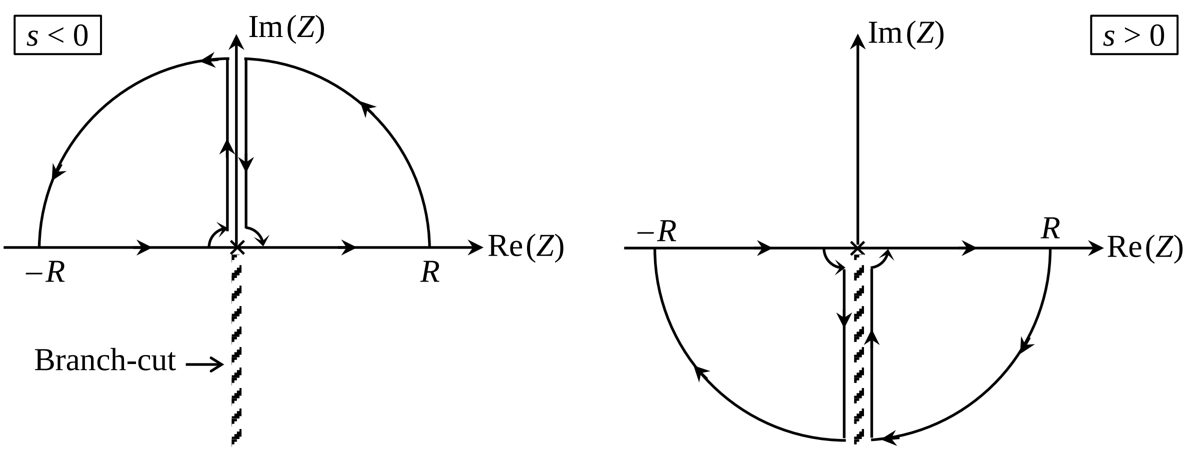
Problem) Use the gamma function $\Gamma(\nu) = \int_0^\infty x^{\nu-1}\exp(-x)\,dx$ to determine the Fourier transform of the function $f(x) = 1/|x|^\nu$, where $0 < \nu < 1$.

Solution) Since the complex function $Z^\nu = |Z|\exp(\mathrm{i}\nu\theta)$ is multi-valued, we must specify a branch-cut in the complex plane. A convenient branch-cut is along the negative imaginary axis, which confines the phase-angle θ to the range $(-90°, 270°]$. The Fourier transform of $f(x)$ may now be written as follows:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}2\pi sx)\,dx \\ &= \int_{-\infty}^{0}[\exp(-\mathrm{i}\pi)x]^{-\nu}\exp(-\mathrm{i}2\pi sx)\,dx + \int_0^\infty x^{-\nu}\exp(-\mathrm{i}2\pi sx)\,dx. \end{aligned}$$

Note that we have set $|x| = \exp(-\mathrm{i}\pi)x$ along the negative x-axis. This choice ensures that, when x is replaced by the complex variable Z, the correct $f(Z)$ is obtained at $Z = |Z|\exp(\mathrm{i}\pi)$.

The two integrals in the above equation must be evaluated on different contours, as shown in the diagram below. The contributions of the large quarter-circles to the corresponding integrals vanish in the limit of large R, in accordance with Jordan's lemma. Also, the small quarter-circles of radius ε do not contribute to the integrals because $dZ = \mathrm{i}\varepsilon\exp(\mathrm{i}\theta)\,d\theta$ approaches zero faster than $Z^\nu = \varepsilon^\nu\exp(\mathrm{i}\nu\theta)$ in the limit when $\varepsilon \to 0$. Consequently, each integral along the real axis is replaced by one along the imaginary axis, as follows.



For $s < 0$, both integrals of $F(s)$ are replaced by integrals along the positive imaginary axis where $Z = \mathrm{i}y$. We will have

$$\begin{aligned} F(s) &= -\mathrm{i}\int_0^\infty[\exp(-\mathrm{i}\pi)\mathrm{i}y]^{-\nu}\exp(2\pi sy)\,dy + \mathrm{i}\int_0^\infty(\mathrm{i}y)^{-\nu}\exp(2\pi sy)\,dy \\ &= -\mathrm{i}\exp(+\tfrac{1}{2}\mathrm{i}\nu\pi)\int_0^\infty y^{-\nu}\exp(2\pi sy)\,dy + \mathrm{i}\exp(-\tfrac{1}{2}\mathrm{i}\nu\pi)\int_0^\infty y^{-\nu}\exp(2\pi sy)\,dy \\ &= 2\sin(\tfrac{1}{2}\nu\pi)\,(2\pi|s|)^{\nu-1}\int_0^\infty \zeta^{-\nu}\exp(-\zeta)\,d\zeta \quad \leftarrow \boxed{\text{Change of variable } \zeta = 2\pi|s|y} \\ &= \frac{2\Gamma(1-\nu)\sin(\tfrac{1}{2}\nu\pi)}{(2\pi|s|)^{1-\nu}}. \end{aligned}$$

For $s > 0$, the two integrals of $F(s)$ are replaced by integrals along the negative imaginary axis, where, in the third quadrant, $Z = \mathrm{i}^3|y|$, whereas in the fourth quadrant $Z = -\mathrm{i}|y|$. Since we are going to choose the range of integration along the negative imaginary axis from zero to $+\infty$, we shall ignore the absolute-value sign and write $|y|$ simply as y. We will have

$$
\begin{aligned}
F(s) &= -\mathrm{i}^3 \int_0^\infty [\exp(-\mathrm{i}\pi)\mathrm{i}^3 y]^{-\nu} \exp(-2\pi s y)\, dy - \mathrm{i} \int_0^\infty (-\mathrm{i}y)^{-\nu} \exp(-2\pi s y)\, dy \\
&= \mathrm{i} \exp(-\tfrac{1}{2}\mathrm{i}\nu\pi) \int_0^\infty y^{-\nu} \exp(-2\pi s y)\, dy - \mathrm{i} \exp(+\tfrac{1}{2}\mathrm{i}\nu\pi) \int_0^\infty y^{-\nu} \exp(-2\pi s y)\, dy \\
&= 2 \sin(\tfrac{1}{2}\nu\pi)\, (2\pi s)^{\nu-1} \int_0^\infty \zeta^{-\nu} \exp(-\zeta)\, d\zeta \qquad \boxed{\begin{array}{c}\text{Change of variable} \\ \zeta = 2\pi s y\end{array}} \\
&= \frac{2\Gamma(1-\nu)\sin(\tfrac{1}{2}\nu\pi)}{(2\pi|s|)^{1-\nu}}.
\end{aligned}
$$

It is seen that the expressions of $F(s)$ thus obtained for positive and negative s are identical. This should not come as a surprise considering that $f(x)$ is a real and even function of x, whose Fourier transform must also be a real and even function of s.