Problem 5)

a) $f_1(z)=\cos z=\tfrac{1}{2}[\exp(\mathrm{i}z)+\exp(-\mathrm{i}z)]=\tfrac{1}{2}[\exp(-y)(\cos x+\mathrm{i}\sin x)+\exp(y)(\cos x-\mathrm{i}\sin x)].$

$$u(x,y)=\tfrac{1}{2}[\exp(y)+\exp(-y)]\cos x; \qquad v(x,y)=-\tfrac{1}{2}[\exp(y)-\exp(-y)]\sin x.$$

$$\frac{\partial u}{\partial x}=-\tfrac{1}{2}[\exp(y)+\exp(-y)]\sin x; \qquad \frac{\partial u}{\partial y}=\tfrac{1}{2}[\exp(y)-\exp(-y)]\cos x.$$

$$\frac{\partial v}{\partial x}=-\tfrac{1}{2}[\exp(y)-\exp(-y)]\cos x; \qquad \frac{\partial v}{\partial y}=-\tfrac{1}{2}[\exp(y)+\exp(-y)]\sin x.$$

Clearly $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\partial v/\partial x$. The function $f_1(z)$ is thus analytic everywhere.

b) $f_2(z)=\dfrac{1}{1+\exp(z)}=\dfrac{1}{1+\exp(x)(\cos y+\mathrm{i}\sin y)}=\dfrac{1+\exp(x)\cos y-\mathrm{i}\exp(x)\sin y}{[1+\exp(x)\cos y]^2+\exp(2x)\sin^2 y}$

$$=\frac{1+\exp(x)\cos y-\mathrm{i}\exp(x)\sin y}{1+\exp(2x)+2\exp(x)\cos y}.$$

$$u(x,y)=\frac{1+\exp(x)\cos y}{1+\exp(2x)+2\exp(x)\cos y}; \qquad v(x,y)=-\frac{\exp(x)\sin y}{1+\exp(2x)+2\exp(x)\cos y}.$$

$$\frac{\partial u}{\partial x}=\frac{\exp(x)\cos y[1+\exp(2x)+2\exp(x)\cos y]-[1+\exp(x)\cos y][2\exp(2x)+2\exp(x)\cos y]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$=-\frac{\exp(x)[\cos y+\exp(2x)\cos y+2\exp(x)]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$\frac{\partial u}{\partial y}=\frac{-\exp(x)\sin y[1+\exp(2x)+2\exp(x)\cos y]+2\exp(x)\sin y[1+\exp(x)\cos y]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$=\frac{\exp(x)\sin y[1-\exp(2x)]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$\frac{\partial v}{\partial x}=-\frac{\exp(x)\sin y[1+\exp(2x)+2\exp(x)\cos y]-\exp(x)\sin y[2\exp(2x)+2\exp(x)\cos y]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$=-\frac{\exp(x)\sin y[1-\exp(2x)]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$\frac{\partial v}{\partial y}=-\frac{\exp(x)\cos y[1+\exp(2x)+2\exp(x)\cos y]+2\exp(x)\sin y[\exp(x)\sin y]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

$$=-\frac{\exp(x)[\cos y+\exp(2x)\cos y+2\exp(x)]}{[1+\exp(2x)+2\exp(x)\cos y]^2}$$

Clearly $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\partial v/\partial x$. The only points where the function $f_2(z)$ is undefined are the roots of the denominator, namely, $\exp(z)=-1=\exp[\mathrm{i}(2n+1)\pi]$, or $z=\mathrm{i}(2n+1)\pi$. Aside from these points, the function is analytic everywhere in the complex plane.