Problem 5) a) The membrane's local slopes along the x and y axes, namely, $\partial_x z(x,y,t)$ and $\partial_y z(x,y,t)$, can be approximated via $\tan\theta \cong \sin\theta$ to yield the vertical component of the force acting on the infinitesimal $\Delta x \times \Delta y$ rectangular section of the membrane, as follows:

$$F_z = T\Delta y[\partial_x z(x+½\Delta x,y,t) - \partial_x z(x-½\Delta x,y,t)] + T\Delta x[\partial_y z(x,y+½\Delta y,t) - \partial_y z(x,y-½\Delta y,t)]. \quad (1)$$

Adding the friction force $-\beta\Delta x\Delta y\partial_t z(x,y,t)$, which acts in opposition to the local velocity, to the above tensile force, then equating the total force with mass $(\rho\Delta x\Delta y)$ times the acceleration $\partial_t^2 z(x,y,t)$ — in accordance with Newton's second law — one arrives at the following equation of motion:

$$v^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] = \frac{\partial^2 z(x,y,t)}{\partial t^2} + \gamma\frac{\partial z(x,y,t)}{\partial t}. \quad (2)$$

b) The boundary conditions on the three sides where the membrane is firmly attached to the drumhead are $z(x=0,y,t) = z(x=L_x,y,t) = z(x,y=0,t) = 0$. On the fourth side, where the membrane is free to vibrate in the z direction, we must have $\partial_y z(x,y=L_y,t) = 0$.

The initial position $z(x,y,t=0)$ and the initial velocity $\partial_t z(x,y,t=0)$ at $t=0$ are known functions of the spatial coordinates (x,y). These constitute the initial conditions for our vibrating membrane.

c) Invoking the method of separation of variables, we write $z(x,y,t) = f(x)g(y)h(t)$. Substitution into the equation of motion then yields

$$v^2\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] = \frac{h''(t)+\gamma h'(t)}{h(t)} = c. \quad (3)$$

On the left-hand-side of the above equation, the first term must equal a constant c_1, that is, $f''(x) = c_1 f(x)$. The boundary conditions at $x=0$ and $x=L_x$ demand that the solution to this equation be $f(x) = \sin(m\pi x/L_x)$, where m is an arbitrary positive integer. Consequently, $c_1 = -(m\pi/L_x)^2$.

As for the second term on the left-hand-side of Eq.(3), we must have $g''(y) = c_2 g(y)$. The boundary conditions now demand that $g(y) = \sin[(n-½)\pi y/L_y]$, where n is another arbitrary positive integer. Consequently, $c_2 = -[(n-½)\pi/L_y]^2$.

The constant c is thus seen to be equal to $(c_1+c_2)v^2 = -\pi^2 v^2[(m/L_x)^2 + (n-½)^2/L_y^2]$. The solutions of the ordinary differential equation $h''(t) + \gamma h'(t) - ch(t) = 0$ are obtained by setting $h(t) = \exp(\eta t)$, which yields $\eta^2 + \gamma\eta - c = 0$. The solutions of this quadratic equation are readily found as $\eta^{\pm} = -½\gamma \pm \sqrt{¼\gamma^2 - \pi^2 v^2[(m/L_x)^2 + (n-½)^2/L_y^2]}$. Depending on the value of the constant inside the radical, the solutions η^+ and η^- may be

i) distinct complex conjugates — i.e., the case of under-damped vibrations;

ii) real and equal — i.e., the case of critically-damped vibrations;

iii) real and distinct — i.e., the case of over-damped vibrations.

The general solution for the time-dependent function is $h(t) = A\exp(\eta^+ t) + B\exp(\eta^- t)$ when $\eta^+ \neq \eta^-$, and $h(t) = A\exp(\eta t) + Bt\exp(\eta t)$ when $\eta^+ = \eta^- = \eta$. In what follows, we shall omit the case of critical damping. The admissible vibrational modes in cases of under-damped and over-damped oscillations are thus given by

$$z_{mn}(x,y,t) = [A_{mn}\exp(\eta_{mn}^{+}t) + B_{mn}\exp(\eta_{mn}^{-}t)]\sin(m\pi x/L_x)\sin[(n-½)\pi y/L_y]. \quad (4)$$

The A_{mn} and B_{mn} for over-damped oscillations are real-valued constant coefficients to be determined by matching the initial conditions at $t = 0$. In the case of under-damped oscillations, where η_{mn}^{+} and η_{mn}^{-} are a pair of complex conjugates, we will have $A_{mn} = B_{mn}^{*}$, in which case the real and imaginary parts of these coefficients are, once again, determined by matching the initial conditions at $t = 0$. A similar procedure, of course, can be followed in cases of critical-damping.

d) The general solution of the wave equation, Eq.(2), subject to the aforementioned boundary conditions is a superposition of all the vibrational modes given in Eq.(4), that is,

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[A_{mn}\exp(\eta_{mn}^{+}t) + B_{mn}\exp(\eta_{mn}^{-}t)]\sin(m\pi x/L_x)\sin[(n-½)\pi y/L_y]. \quad (5)$$

The unknown coefficients A_{mn} and B_{mn} must be obtained from the initial conditions. Upon expanding $z(x,y,t=0)$ and $\partial_t z(x,y,t=0)$ in their respective Fourier series, then matching the corresponding Fourier coefficients with those given by (or derived from) Eq.(5), the general solution $z(x,y,t)$ for all times $t \geq 0$ will be uniquely identified.