Problem 5) a) $g(x)a_0(x)f''(x) + g(x)a_1(x)f'(x) + [g(x)a_2(x) + \lambda g(x)a_3(x)]f(x) = 0.$

Standard Sturm-Liouville form: $\frac{d}{dx}[p(x)f'(x)] + [q(x) + \lambda r(x)]f(x) = 0 \Rightarrow$

$p(x)f''(x) + p'(x)f'(x) + [q(x) + \lambda r(x)]f(x) = 0 \Rightarrow$

$g(x)a_0(x) = p(x)$; $g(x)a_1(x) = p'(x)$; $g(x)a_2(x) = q(x)$; $g(x)a_3(x) = r(x)$.

$\Rightarrow \frac{p'(x)}{p(x)} = \frac{a_1(x)}{a_0(x)} \Rightarrow \ln p(x) = \int \frac{a_1(x)}{a_0(x)}dx \Rightarrow p(x) = \exp\left[\int \frac{a_1(x)}{a_0(x)}dx\right].$

$\Rightarrow g(x) = \frac{p(x)}{a_0(x)} \Rightarrow q(x) = p(x)\frac{a_2(x)}{a_0(x)}$; $r(x) = p(x)\frac{a_3(x)}{a_0(x)}.$

b) $f''(x) + \cot g(x)f'(x) + \lambda f(x) = 0.$

Using the result of part (a), we write: $p(x) = e^{\int \frac{a_1(x)}{a_0(x)}dx} = e^{\int \cot g(x)\,dx} \Rightarrow$

$p(x) = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln(\sin x)} = \sin x.$ Therefore, $q(x) = 0$

and $r(x) = \sin(x).$ The standard form of the equation thus becomes:

$$\frac{d}{dx}[\sin x\, f'(x)] + \lambda \sin x\, f(x) = 0.$$

$x f''(x) + (1-x)f'(x) + (x^2+\lambda)f(x) = 0.$

Using the result of part (a), we write: $p(x) = e^{\int \frac{a_1(x)}{a_0(x)}dx} = e^{\int (\frac{1}{x}-1)dx} \Rightarrow$

$p(x) = e^{\ln x - x} = x e^{-x}.$ Therefore, $q(x) = x e^{-x}\frac{x^2}{x} = x^2 e^{-x}$ and

$r(x) = x e^{-x}\frac{1}{x} = e^{-x}.$ The standard form of the equation thus becomes:

$$\frac{d}{dx}[x e^{-x} f'(x)] + e^{-x}(x^2+\lambda)f(x) = 0.$$