Problem 4) One may write $f(x)$ in several different ways using comb functions, delta-functions, and various combinations thereof. Here are three different ways of expressing the same function:

a) $f(x)=\frac{1}{3}\text{comb}(\frac{x}{3})+\frac{1}{6}\text{comb}(\frac{x-1}{3}) \rightarrow F(s)=\text{comb}(3s)+\frac{1}{2}\exp(-i2\pi s)\,\text{comb}(3s).$

scaling and shift theorems

Now, the function $F(s)=[1+\frac{1}{2}\exp(-i2\pi s)]\text{comb}(3s)$ is periodic with a period of 1.0, because $\exp(-i2\pi s)$ has period 1.0 along the s-axis, while $\text{comb}(3s)$ has a period of 1/3. We thus need to identify only the magnitudes of the three delta-functions located at $s=0$, 1/3, and 2/3. Noting that each of the delta-functions comprising $\text{comb}(3s)$ has an amplitude of 1/3, the general formula for the magnitude of the delta-function located at $s_n = n/3$ is $\frac{1}{3}[1+\frac{1}{2}\exp(-i2\pi n/3)]$. Thus the delta-function located at $s=s_0$ has amplitude ½, that at $s=s_1$ has amplitude $¼(1-i/\sqrt{3})$, and that at $s=s_2$ has amplitude $¼(1+i/\sqrt{3})$.

b) $f(x)=[\delta(x)+\frac{1}{2}\delta(x-1)]*\frac{1}{3}\text{comb}(\frac{x}{3}) \rightarrow F(s)=[1+\frac{1}{2}\exp(-i2\pi s)]\text{comb}(3s).$

c) $f(x)=\text{comb}(x)-\frac{1}{6}\text{comb}(\frac{x-1}{3})-\frac{1}{3}\text{comb}(\frac{x-2}{3})$. In this expression, the first comb function places a unit-magnitude delta-function at $x=0,\pm1,\pm2$, etc. The second term reduces from 1 to ½ the amplitude of the delta-functions at $x=-5,-2,1,4,7$, and so on. The third term eliminates the delta-functions at $x=-4,-1,2,5,8$, etc. Straightforward Fourier transformation with the aid of scaling and shift theorems then yields:

$$F(s)=\text{comb}(s)-\tfrac{1}{2}\exp(-i2\pi s)\,\text{comb}(3s)-\exp(-i4\pi s)\,\text{comb}(3s).$$

As before, this is a periodic function with period 3. The magnitudes of the delta-functions, located at $s_0=0$, $s_1=1/3$ and $s_2=2/3$ are found from the above expression to be ½, $¼(1-i/\sqrt{3})$, and $¼(1+i/\sqrt{3})$, confirming the preceding results.