Problem 4) a) $z_1(t) = \int_{-\infty}^{\infty} Z_1(s)\, e^{+i2\pi st}\, ds \Rightarrow z_1'(t) = \int_{-\infty}^{\infty} i2\pi s\, Z_1(s)\, e^{+i2\pi st}\, ds$

$\Rightarrow z_1''(t) = \int_{-\infty}^{\infty} (i2\pi s)^2 Z_1(s)\, e^{+i2\pi st}\, ds.$

Therefore, $\mathcal{F}\{z_1(t)\} = Z_1(s)$; $\mathcal{F}\{z_1'(t)\} = i2\pi s Z_1(s)$; $\mathcal{F}\{z_1''(t)\} = -4\pi^2 s^2 Z_1(s)$.

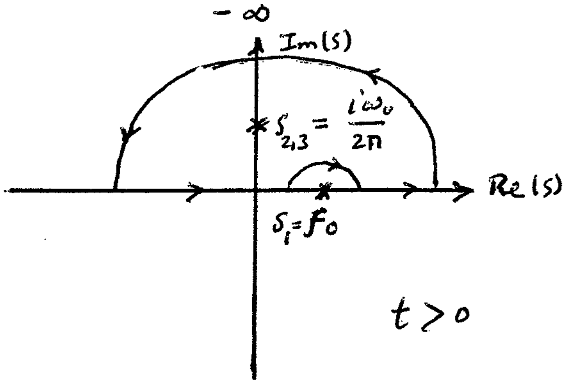
Since $f(t) = F_0\, \mathrm{Step}(t) \sin(2\pi f_0 t) = \mathrm{Im}\{F_0\, \mathrm{Step}(t)\, e^{i2\pi f_0 t}\}$, we solve the problem for the excitation function $F_0\, \mathrm{Step}(t)\, e^{i2\pi f_0 t}$, then, in the end, take the imaginary part of $z_1(t)$.

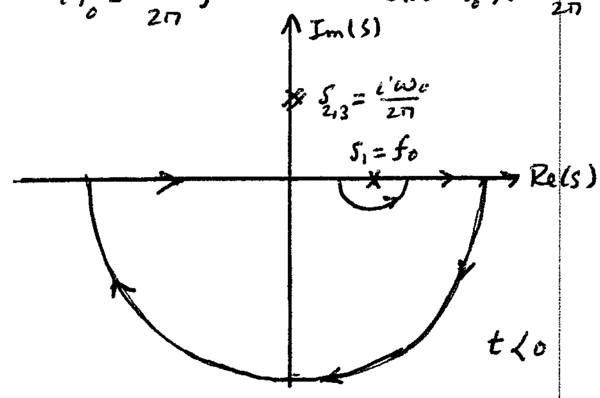
$$\mathcal{F}\{F_0\, \mathrm{Step}(t)\, e^{i2\pi f_0 t}\} = F_0\left[\tfrac{1}{2}\delta(s) - \frac{i}{2\pi s}\right] * \delta(s-f_0) = \frac{F_0}{2}\delta(s-f_0) - \frac{iF_0}{2\pi(s-f_0)}.$$

The F.T. of the differential equation thus yields:

$$Z_1(s) = \frac{\frac{F_0}{2m}\delta(s-f_0) - \frac{iF_0}{2\pi m(s-f_0)}}{-4\pi^2 s^2 + i4\pi s\omega_0 + \omega_0^2} = \frac{F_0\, \delta(s-f_0)}{-8\pi^2 m\left(f_0 - \frac{i\omega_0}{2\pi}\right)^2} + \frac{iF_0}{8\pi^3 m (s-f_0)\left(s - \frac{i\omega_0}{2\pi}\right)^2}.$$

Therefore, $$z_1(t) = \int_{-\infty}^{\infty} Z_1(s) e^{+i2\pi st}\, ds = -\frac{F_0}{8\pi^2 m}\, \frac{e^{i2\pi f_0 t}}{\left(f_0 - \frac{i\omega_0}{2\pi}\right)^2} + \frac{iF_0}{8\pi^3 m}\int_{-\infty}^{\infty} \frac{e^{i2\pi st}}{(s-f_0)\left(s-\frac{i\omega_0}{2\pi}\right)^2}\, ds.$$





For $t < 0$, Jordan's lemma dictates the use of the lower-half-plane. The only relevant pole here is $s_1 = f_0$ and, since it's on-axis, only half of the residue will contribute to the integral. The sign of the residue is negative because of the clockwise direction of travel on the small semi-circle.

We thus have:

$$t<0 \; : \; z_1(t) = -\frac{F_0}{8\pi^2 m}\,\frac{e^{i2\pi f_0 t}}{\left(f_0 - \frac{i\omega_0}{2\pi}\right)^2} - \frac{1}{2}(2\pi i)\frac{iF_0}{8\pi^3 m}\,\frac{e^{i2\pi f_0 t}}{\left(f_0-\frac{i\omega_0}{2\pi}\right)^2} = 0 \checkmark$$

For $t>0$, Jordon's lemma dictates the use of the upper-half-plane. Here the pole at $s_1 = f_0$ contributes one-half of its residue (because it is on-axis), but the sign of its contribution will be <u>positive</u>, because of the counter-clockwise direction of travel on the small semi-circle.

The contribution of the double-pole (or second-order pole) at $s = \frac{i\omega_0}{2\pi}$ is given by the derivative of $\frac{e^{i2\pi st}}{s-f_0}$ evaluated at $s = \frac{i\omega_0}{2\pi}$, that is,

$$\frac{d}{ds}\left(\frac{e^{i2\pi st}}{s-f_0}\right)\Bigg|_{s=\frac{i\omega_0}{2\pi}} = \frac{i2\pi t\, e^{i2\pi\left(\frac{i\omega_0}{2\pi}\right)t}}{\frac{i\omega_0}{2\pi}-f_0} - \frac{e^{i2\pi\left(\frac{i\omega_0}{2\pi}\right)t}}{\left(\frac{i\omega_0}{2\pi}-f_0\right)^2}$$

$$= \frac{4\pi^2 t\, e^{-\omega_0 t}}{\omega_0 + i2\pi f_0} + \frac{4\pi^2 e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)^2}.$$

Therefore,

$$z_1(t) = -\frac{F_0}{8\pi^2 m}\,\frac{e^{i2\pi f_0 t}}{\left(f_0-\frac{i\omega_0}{2\pi}\right)^2} + \frac{1}{2}(2\pi i)\frac{iF_0}{8\pi^3 m}\,\frac{e^{i2\pi f_0 t}}{\left(f_0-\frac{i\omega_0}{2\pi}\right)^2}$$

$$+ (2\pi i)\frac{iF_0}{8\pi^3 m}\left\{\frac{4\pi^2 t e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)} + \frac{4\pi^2 e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)^2}\right\}$$

$$= \frac{F_0 e^{i2\pi f_0 t}}{m(\omega_0 + i2\pi f_0)^2} - \frac{F_0 e^{-\omega_0 t}}{m(\omega_0+i2\pi f_0)^2}\left[1 + (\omega_0 + i2\pi f_0)t\right]$$

$$= \frac{F_0}{m(\omega_0^2 + 4\pi^2 f_0^2)\, e^{2i\tan^{-1}(2\pi f_0/\omega_0)}}\left[e^{i2\pi f_0 t} - e^{-\omega_0 t} - \sqrt{\omega_0^2 + 4\pi^2 f_0^2}\; e^{i\tan^{-1}(2\pi f_0/\omega_0)} \times t e^{-\omega_0 t}\right]$$

Note that $z_1(0^+) = 0$ and also $z_1'(0^+) = 0$, as required by the continuity conditions for a second-order differential equation.

Next, we take the imaginary part of $z_1(t)$ as the final solution, because the excitation function is the imaginary part of $F_0 \, \mathrm{Step}(t) \, e^{i2\pi f_0 t}$. We'll have:

$$z_1(t) = \frac{F_0 \, \mathrm{Step}(t)}{m(\omega_0^2 + 4\pi^2 f_0^2)} \Big\{ \sin\left[2\pi f_0 t - 2\tan^{-1}(2\pi f_0/\omega_0)\right] + \sin\left[2\tan^{-1}(2\pi f_0/\omega_0)\right] e^{-\omega_0 t}$$

$$+ \sqrt{\omega_0^2 + 4\pi^2 f_0^2} \, \sin\left[\tan^{-1}(2\pi f_0/\omega_0)\right] t e^{-\omega_0 t} \Big\} \Rightarrow$$

$$z_1(t) = \frac{F_0 \, \mathrm{Step}(t)}{m(\omega_0^2 + 4\pi^2 f_0^2)} \left\{ \sin\left[2\pi f_0 t - 2\tan^{-1}\left(\frac{2\pi f_0}{\omega_0}\right)\right] + \frac{4\pi f_0 \omega_0 e^{-\omega_0 t}}{\omega_0^2 + 4\pi^2 f_0^2} + 2\pi f_0 t e^{-\omega_0 t} \right\}$$

b) Because of damping, the natural frequency ω_0 may be used for excitation without causing the excitation amplitude to go to infinity. Thus, when $2\pi f_0 = \omega_0$, all we need to do is substitute $2\pi f_0$ with ω_0 in the above expression. We'll have:

$$z_1(t) = \frac{F_0 \, \mathrm{Step}(t)}{2m\omega_0^2} \left\{ \sin\left(\omega_0 t - \frac{\pi}{2}\right) + \frac{2\omega_0^2 e^{-\omega_0 t}}{2\omega_0^2} + \omega_0 t e^{-\omega_0 t} \right\} \Rightarrow$$

$$z_1(t) = \frac{F_0 \, \mathrm{step}(t)}{2m\omega_0^2} \left[(1 + \omega_0 t) e^{-\omega_0 t} - \cos(\omega_0 t) \right].$$

Once again, it is seen that $z_1(0^+) = z_1'(0^+) = 0$, as required by the continuity condition.