Problem 4) Writing the separable solution as $\psi(\boldsymbol{r},t)=f(r)g(\varphi)h(z)p(t)$, upon substitution into the wave equation and division by ψ, we find

$$v^2\left[\frac{f''(r)}{f(r)}+\frac{f'(r)}{rf(r)}+\frac{g''(\varphi)}{r^2g(\varphi)}+\frac{h''(z)}{h(z)}\right]=\frac{p''(t)+\gamma p'(t)}{p(t)}. \quad (1)$$

Both sides of the above equation must now be equated to a negative constant $-c^2$, because otherwise one of the solutions for $p(t)$ will grow indefinitely with time, which is physically inadmissible. The left-hand side of Eq.(1) can be a constant only if its various terms that depend on r, φ, and z are separately equal to constants. We thus have

$$\frac{g''(\varphi)}{g(\varphi)}=-m^2 \quad\rightarrow\quad g(\varphi)=A_1\cos(m\varphi)+A_2\sin(m\varphi) \quad\rightarrow\quad g(\varphi)=A\cos(m\varphi+\varphi_0). \quad (2)$$

$$\frac{h''(z)}{h(z)}=-k_z^2 \quad\rightarrow\quad h(z)=B_1\sin(k_zz)+B_2\cos(k_zz) \quad\rightarrow\quad h(z)=B\sin(\ell\pi z/L). \quad (3)$$

$$\frac{f''(r)}{f(r)}+\frac{f'(r)}{rf(r)}-\frac{m^2}{r^2}-k_z^2=-(c/v)^2 \quad\rightarrow\quad r^2f''(r)+rf'(r)+(k_r^2r^2-m^2)f(r)=0. \quad (4)$$

In the above equations, we have introduced the integers m and ℓ as the mode indices in the azimuthal and vertical directions φ and z, respectively. We have also defined in Eq.(4) the new parameter $k_r=\sqrt{(c/v)^2-k_z^2}=\sqrt{(c/v)^2-(\ell\pi/L)^2}$. The solutions to the Bessel equation are

$$f(r)=C_1J_m(k_rr)+C_2Y_m(k_rr). \quad (5)$$

Since the volume of interest contains the z-axis, for which $r=0$, the term containing a Bessel function of the second kind $Y_m(\cdot)$ must vanish, that is $C_2=0$. Given that $\psi(\boldsymbol{r},t)=0$ at $r=R$, we must have $k_rR=\rho_{mn}$, where ρ_{mn} is the n^{th} zero of $J_m(\rho)$. Consequently,

$$\left(\frac{c}{v}\right)^2-\left(\frac{\ell\pi}{L}\right)^2=\left(\frac{\rho_{mn}}{R}\right)^2 \quad\rightarrow\quad c^2=\left[\left(\frac{v\rho_{mn}}{R}\right)^2+\left(\frac{v\ell\pi}{L}\right)^2\right]. \quad (6)$$

The time-dependent factor $p(t)$ is thus seen to be the solution of the following equation:

$$p''(t)+\gamma p'(t)+c^2p(t)=0. \quad (7)$$

The solutions of Eq.(7) are in the form of $\exp(\eta t)$, where $\eta^2+\gamma\eta+c^2=0$. consequently,

$$\eta_\pm=-(\gamma/2)\pm\sqrt{(\gamma/2)^2-c^2}. \quad (8)$$

Had we chosen the initial separation constant to be positive (i.e., c^2 rather than $-c^2$), Eq.(8) would have yielded a positive value for η_+, which would have been physically untenable. With a negative separation constant, the two values of η will be either real and negative (over-damped), real, negative and equal (critically-damped), or complex conjugates with a negative real part (under-damped). The general solution of Eq.(7) thus acquires one of the following forms:

$$p(t)=\begin{cases}D_1\exp(\eta_+t)+D_2\exp(\eta_-t)\,; & \text{(overdamped)}\\ D_1\exp(-½\gamma t)+D_2t\exp(-½\gamma t)\,; & \text{(critically}-\text{damped)}\\ D\exp(-½\gamma t)\cos\left[\sqrt{(v\rho_{mn}/R)^2+(v\ell\pi/L)^2-(\gamma/2)^2}\,t+\chi_0\right]; & \text{(unerdamped).}\end{cases} \quad (9)$$

Thus the general solution in the underdamped case, for instance, is written as follows:

$$\psi(\boldsymbol{r},t) = \sum_m \sum_n \sum_\ell C_{mn\ell}\, J_m\left(\frac{\rho_{mn} r}{R}\right) \cos(m\varphi + \varphi_{mn\ell}) \sin(\ell\pi z/L) \times \exp(-\tfrac{1}{2}\gamma t)\; \cos(\omega_{mn\ell} t + \chi_{mn\ell}). \qquad (10)$$

The unknown parameters $C_{mn\ell}$, $\varphi_{mn\ell}$, and $\chi_{mn\ell}$ must be determined from the initial conditions at $t = 0$.
