Problem 3) a) $\mathcal{F}\{\text{sign}(x)\} = \mathcal{F}\{\text{step}(x) - \text{step}(-x)\}$

linear superposition theorem → $= \mathcal{F}\{\text{step}(x)\} - \mathcal{F}\{\text{step}(-x)\}$

scaling theorem ($\alpha = -1$) → $= [½\delta(s) - \text{i}/(2\pi s)] - [½\delta(-s) - \text{i}/(-2\pi s)]$

δ-function is even: $\delta(s) = \delta(-s)$ → $= ½[\delta(s) - \delta(s)] - \text{i}/(\pi s)$

$= -\text{i}/(\pi s)$.

b) On the vertical legs of the contour, $z = \pm L + \text{i}y$, with $0 \le y \le N$. Considering that

$$\tanh(\pi z) = \frac{\sinh(\pi z)}{\cosh(\pi z)} = \frac{\exp[\pi(\pm L+\text{i}y)] - \exp[-\pi(\pm L+\text{i}y)]}{\exp[\pi(\pm L+\text{i}y)] + \exp[-\pi(\pm L+\text{i}y)]} \to \pm 1 \quad \text{when } L \to \infty,$$

we will have, along the vertical leg on the right-hand side,

$$\int_{z=L+\text{i}0}^{L+\text{i}N} \tanh(\pi z)\, e^{-\text{i}2\pi zs} \text{d}z \underset{(L\to\infty)}{\longrightarrow} \text{i}\int_0^N e^{-\text{i}2\pi(L+\text{i}y)s}\text{d}y = \frac{\text{i}e^{-\text{i}2\pi Ls}}{2\pi s}(e^{2\pi Ns} - 1).$$

Similarly, along the vertical leg on the left-hand side, we find

$$\int_{z=-L+\text{i}0}^{-L+\text{i}N} \tanh(\pi z)\, e^{-\text{i}2\pi zs} \text{d}z \underset{(L\to\infty)}{\longrightarrow} \frac{\text{i}e^{\text{i}2\pi Ls}}{2\pi s}(e^{2\pi Ns} - 1).$$

Thus, the contribution of the two vertical legs to the overall loop integral is given by

$$(e^{2\pi Ns} - 1)\frac{\sin(2\pi Ls)}{\pi s}.$$

When $L \to \infty$, this becomes a highly oscillatory function of s with no significant physical effects, which may therefore be ignored.

The poles of the integrand are where the denominator $\cosh(\pi z)$ of $\tanh(\pi z)$ vanishes. These are readily found to be $z_n = \text{i}(n - ½)$. A Taylor series expansion of $\cosh(\pi z)$ around z_n yields

In the vicinity of z_n: $$\tanh(\pi z)\, e^{-\text{i}2\pi zs} \cong \frac{\sinh(\pi z_n)}{\cosh(\pi z_n) + \pi \sinh(\pi z_n)(z - z_n)} e^{-\text{i}2\pi z_n s}.$$ ($\cosh(\pi z_n) \to 0$)

The residue at each pole z_n is now seen to be

$$\frac{\sinh(\pi z_n)}{\pi \sinh(\pi z_n)} e^{-\text{i}2\pi z_n s} = \frac{e^{2\pi(n-½)s}}{\pi}.$$

Cauchy's theorem, applied to the upper and lower legs of the contour, now yields

$$\int_{-L}^{L} \tanh(\pi x)\, e^{-\text{i}2\pi sx}\text{d}x + \int_{L+\text{i}N}^{-L+\text{i}N} \tanh(\pi x + \text{i}N\pi)\, e^{-\text{i}2\pi s(x+\text{i}N)}\text{d}x = \text{i}2\pi \sum_{n=1}^{N} \frac{e^{2\pi(n-½)s}}{\pi}$$

$$\to (1 - e^{2\pi Ns})\int_{-\infty}^{\infty} \tanh(\pi x)\, e^{-\text{i}2\pi sx}\text{d}x = \text{i}2e^{\pi s}(1 - e^{2\pi Ns})/(1 - e^{2\pi s})$$ ← geometric series

$$\to \int_{-\infty}^{\infty} \tanh(\pi x)\, e^{-\text{i}2\pi sx}\text{d}x = \text{i}\left(\frac{2e^{\pi s}}{1 - e^{2\pi s}}\right) = -\text{i}/\sinh(\pi s).$$

As expected, the final result is independent of N, indicating that any contour that contains 1 or 2 or 3 or any other number of poles would yield exactly the same result.

c) In the limit when $\alpha \to 0$, depending on whether x is positive or negative, the argument $\pi x/\alpha$ of $\tanh(\pi x/\alpha)$ goes to $\pm\infty$. Given that $\tanh(\pm\infty) = \pm 1$, it is readily observed that $\tanh(\pi x/\alpha)$ must approach $\mathrm{sign}(x)$. Applying the scaling theorem to the final result of part (b), we find

$$\mathcal{F}\{\tanh(\pi x/\alpha)\} = -\mathrm{i}\alpha/\sinh(\pi\alpha s).$$

In the limit when $\alpha \to 0$ the function $\tanh(\pi x/\alpha)$ approaches $\mathrm{sign}(x)$ and $\sinh(\pi\alpha s) \to \pi\alpha s$. Consequently, the Fourier transform of $\mathrm{sign}(x)$ is seen to be $-\mathrm{i}\alpha/(\pi\alpha s) = -\mathrm{i}/(\pi s)$.
