Problem 2) a) Let the Fourier transform of $f(x)$ be $F(s) = \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}2\pi sx)\,\mathrm{d}x$. Then $f(x) = \int_{-\infty}^{\infty} F(s)\exp(\mathrm{i}2\pi sx)\,\mathrm{d}s$, and $\mathcal{F}\{f'(x)\} = \mathrm{i}2\pi sF(s)$. The Fourier transform of the differential equation may thus be written as

$$\mathrm{i}2\pi sF(s) + \eta F(s) = \mathrm{Sinc}(s) \qquad \rightarrow \qquad F(s) = \frac{\sin(\pi s)}{\pi s\,(\mathrm{i}2\pi s + \eta)}. \tag{1}$$

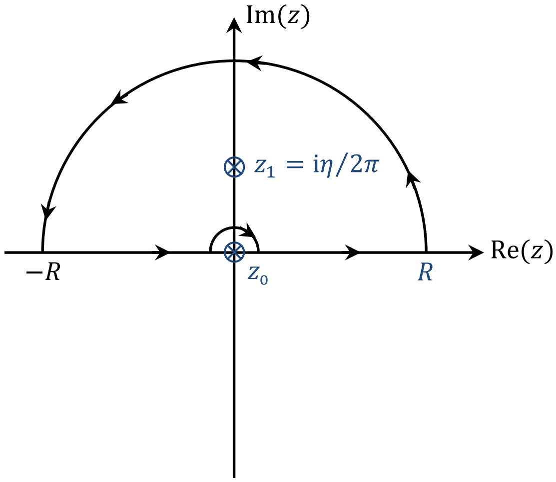
The solution of the differential equation may now be obtained by inverse Fourier transforming the above $F(s)$, as follows:

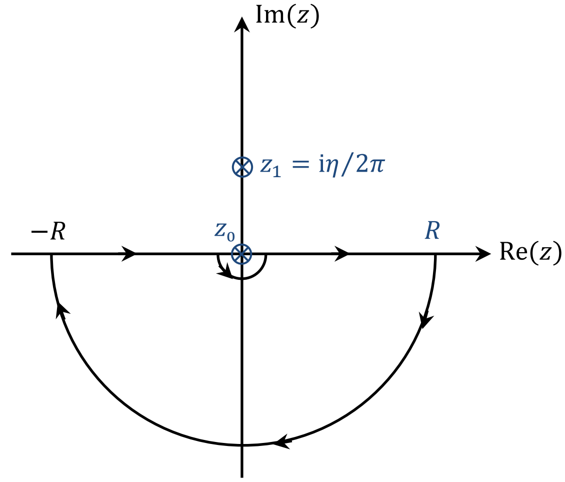
$$f(x) = \mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} \frac{\sin(\pi s)}{\pi s\,(\mathrm{i}2\pi s + \eta)} \exp(\mathrm{i}2\pi sx)\,\mathrm{d}s$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\pi s) - \exp(-\mathrm{i}\pi s)}{s[s - \mathrm{i}(\eta/2\pi)]} \exp(\mathrm{i}2\pi sx)\,\mathrm{d}s$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}2\pi(x+½)s]}{s[s - \mathrm{i}(\eta/2\pi)]}\,\mathrm{d}s + \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}2\pi(x-½)s]}{s[s - \mathrm{i}(\eta/2\pi)]}\,\mathrm{d}s. \tag{2}$$

The integrands on the right-hand-side of Eq.(2) have two poles, one at $z_0 = 0$, the other at $z_1 = \mathrm{i}\eta/2\pi$, as shown in the figures below. Depending on the value of x, the integration contour may be in the upper- or lower-half of the complex plane. The contribution of the large semi-circle to the loop integral vanishes when its radius R goes to infinity (Jordan's lemma). As for the pole at z_0, only one-half of its residue must be taken into account because this pole is located directly on the x-axis.





Both integrals must be evaluated in the lower-half of the complex plane when $x < -½$. Thus

$$\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}2\pi(x\pm½)s]}{s[s - \mathrm{i}(\eta/2\pi)]}\,\mathrm{d}s = -\mathrm{i}\pi\,\frac{\exp[\mathrm{i}2\pi(x\pm½)z_0]}{z_0 - \mathrm{i}(\eta/2\pi)} = \frac{2\pi^2}{\eta}. \tag{3}$$

The same result continues to apply to the second integral for $x < ½$ as well. If $x > -½$, the first integral must be evaluated in the upper-half plane, as follows:

$$\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}2\pi(x+½)s]}{s[s-\mathrm{i}(\eta/2\pi)]}\,\mathrm{d}s = \mathrm{i}2\pi \frac{\exp[\mathrm{i}2\pi(x+½)z_1]}{z_1} + \mathrm{i}\pi \frac{\exp[\mathrm{i}2\pi(x+½)z_0]}{z_0 - \mathrm{i}(\eta/2\pi)}$$

$$= (4\pi^2/\eta)\{\exp[-\eta(x+½)] - ½\}. \quad (4)$$

Finally, if $x > ½$, the second integral must also be evaluated in the upper-half-plane, that is,

$$\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}2\pi(x-½)s]}{s[s-\mathrm{i}(\eta/2\pi)]}\,\mathrm{d}s = \mathrm{i}2\pi \frac{\exp[\mathrm{i}2\pi(x-½)z_1]}{z_1} + \mathrm{i}\pi \frac{\exp[\mathrm{i}2\pi(x-½)z_0]}{z_0 - \mathrm{i}(\eta/2\pi)}$$

$$= (4\pi^2/\eta)\{\exp[-\eta(x-½)] - ½\}. \quad (5)$$

The complete solution is now obtained from Eq.(2) upon substitution from Eqs.(3)-(5), as follows:

$$f(x) = \begin{cases} 0; & x < -½, \\ \{1 - \exp[-\eta(x+½)]\}/\eta\,; & -½ < x < ½, \\ [\exp(\eta/2) - \exp(-\eta/2)]\exp(-\eta x)/\eta\,; & x > ½. \end{cases} \quad (6)$$

b) The function $f(x)$ is continuous at $x = -½$, where $f(x^-) = f(x^+) = 0$, and also at $x = ½$, where $f(x^-) = f(x^+) = [1 - \exp(-\eta)]/\eta$. Any discontinuity in $f(x)$ would have been unacceptable, because the original differential equation contains $f'(x)$ on the left-hand side, but no corresponding delta-functions on the right-hand side. Note also that $f(x)$ approaches zero as $x \to \infty$, all of which in keeping with one's expectations from the solution of the differential equation.

c) A plot of $f(x)$ is shown below.