Problem 2) a) $F_L(s)=\int_{-\infty}^{\infty} f_L(x)\exp(-\mathrm{i}2\pi sx)\,dx$

$$=-\int_{-L}^{0}[1+(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx+\int_{0}^{L}[1-(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx$$

Change of variable: $x\to -x$ →

$$=-\int_{0}^{L}[1-(x/L)]\exp(+\mathrm{i}2\pi sx)\,dx+\int_{0}^{L}[1-(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx$$

$$=-2\mathrm{i}\int_{0}^{L}[1-(x/L)]\sin(2\pi sx)\,dx$$

Integration by parts →

$$=2\mathrm{i}\,\frac{[1-(x/L)]\cos(2\pi sx)}{2\pi s}\Big|_0^L+2\mathrm{i}\int_0^L\frac{(1/L)\cos(2\pi sx)}{2\pi s}\,dx$$

$$=-\frac{2\mathrm{i}}{2\pi s}+2\mathrm{i}\,\frac{\sin(2\pi sx)}{(2\pi s)^2L}\Big|_0^L=\frac{1}{\mathrm{i}\pi s}-\frac{\sin(2\pi sL)}{\mathrm{i}2(\pi s)^2L}=\frac{1-\mathrm{sinc}(2sL)}{\mathrm{i}\pi s}.$$

b) $\mathrm{Lim}_{L\to\infty} f_L(x)=\mathrm{Sgn}(x).$

c) Considering that $\mathrm{sinc}(2sL)\to 0$ when $L\to\infty$, we find: $\mathrm{Lim}_{L\to\infty} F_L(s)=1/(\mathrm{i}\pi s)$.

Note: When $s=0$, the limit of $\mathrm{sinc}(2sL)=\mathrm{sinc}(0)=1$, independent of L. Consequently $\mathrm{Lim}_{L\to\infty} F_L(0)=0$.

d) $$f(x)=\int_{-\infty}^{\infty}F(s)\exp(\mathrm{i}2\pi xs)\,ds=\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}2\pi xs)}{\mathrm{i}\pi s}\,ds.$$

Contours of integration in the complex plane are shown in the diagrams below. On the large semi-circle, the exponent of $\exp(\mathrm{i}2\pi xZ)$ must acquire a negative real part in order to satisfy Jordan's lemma. Thus, when x is positive (negative), the integration must be carried out in the upper (lower) half-plane. In each case, the contribution of the large semi-circle to the integral vanishes, and the integral over the real-axis becomes equal to that over the small semi-circle, taken counterclockwise when $x>0$, and clockwise when $x<0$.

$$f(x)=\int_{\substack{\text{small}\\ \text{semi-circle}}}\frac{\exp(\mathrm{i}2\pi xZ)}{\mathrm{i}\pi Z}\,dZ=\lim_{\varepsilon\to 0}\int_{\theta=0}^{\pm\pi}\frac{\exp[\mathrm{i}2\pi x\varepsilon\exp(\mathrm{i}\theta)]}{\mathrm{i}\pi\varepsilon\exp(\mathrm{i}\theta)}\,\mathrm{i}\varepsilon\exp(\mathrm{i}\theta)\,d\theta=\begin{cases}+\pi/\pi\,, & x>0;\\ -\pi/\pi\,, & x<0.\end{cases}$$

Clearly then $f(x)=\mathrm{Sgn}(x)$, as expected.