Problem 1)

a)
$$zJ_{n-1}(z)+zJ_{n+1}(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} z\{e^{\mathrm{i}[z\sin\theta-(n-1)\theta]}+e^{\mathrm{i}[z\sin\theta-(n+1)\theta]}\}\mathrm{d}\theta$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} z(e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta})e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$
$$=\frac{2}{2\pi}\int_{-\pi}^{\pi} z\cos\theta\, e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$

add and subtract n →
$$=\frac{2}{2\pi}\int_{-\pi}^{\pi}(z\cos\theta-n+n)e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$
$$=\frac{1}{\mathrm{i}\pi}e^{\mathrm{i}(z\sin\theta-n\theta)}\Big|_{\theta=-\pi}^{\pi}+\frac{2n}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$

$\sin\pi=0$ and $e^{\pm in\pi}=(-1)^n$ →
$$=\frac{e^{\mathrm{i}(z\sin\pi-n\pi)}-e^{-\mathrm{i}(z\sin\pi-n\pi)}}{\mathrm{i}\pi}+2nJ_n(z)=2nJ_n(z).$$

b)
$$J_{n-1}(z)-J_{n+1}(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{e^{\mathrm{i}[z\sin\theta-(n-1)\theta]}-e^{\mathrm{i}[z\sin\theta-(n+1)\theta]}\}\mathrm{d}\theta$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta})e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$
$$=\frac{2}{2\pi}\int_{-\pi}^{\pi}\mathrm{i}\sin\theta\, e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta$$
$$=\frac{2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}}{\mathrm{d}z}e^{\mathrm{i}(z\sin\theta-n\theta)}\mathrm{d}\theta=2\frac{\mathrm{d}}{\mathrm{d}z}J_n(z).$$

Digression: These functional relations are general and remain valid for all Bessel functions of the first as well as second kind, arbitrary order ν. Proofs can be constructed along the same lines as above, using the following integral representations of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$:

$$J_\nu(z)=\frac{1}{\pi}\int_0^{\pi}\cos(\nu\theta-z\sin\theta)\,\mathrm{d}\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^{\infty}e^{-\nu t-z\sinh t}\mathrm{d}t,\quad [\mathrm{Re}(z)>0].$$ (G&R **8.411**-13)

$$Y_\nu(z)=\frac{1}{\pi}\int_0^{\pi}\sin(z\sin\theta-\nu\theta)\,\mathrm{d}\theta-\frac{1}{\pi}\int_0^{\infty}(e^{\nu t}+e^{-\nu t}\cos\nu\pi)e^{-z\sinh t}\mathrm{d}t,\quad [\mathrm{Re}(z)>0].$$ (G&R **8.415**-4)