Opti 503A **Midterm Exam** (3/17/2021) **Time: 75 minutes**

Please write your name and ID number on the first page before scanning/photographing the pages. Answer all the questions.

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7 pts **Problem 1**) Without using a calculator or a computer program, sketch a rough plot of the function $f(x) = x^{-1}\ln x$ versus x for positive values of x. Identify the maxima, minima, and inflection points, if any, of the function. Evaluate $\int_1^{x_0} x^{-1}\ln x \,\mathrm{d}x$ for arbitrary values of $x_0 \geq 0$.

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6 pts **Problem 2**) Use proof by induction to show that $1^3 + 2^3 + 3^3 + \cdots + N^3 = [N(N+1)/2]^2$.

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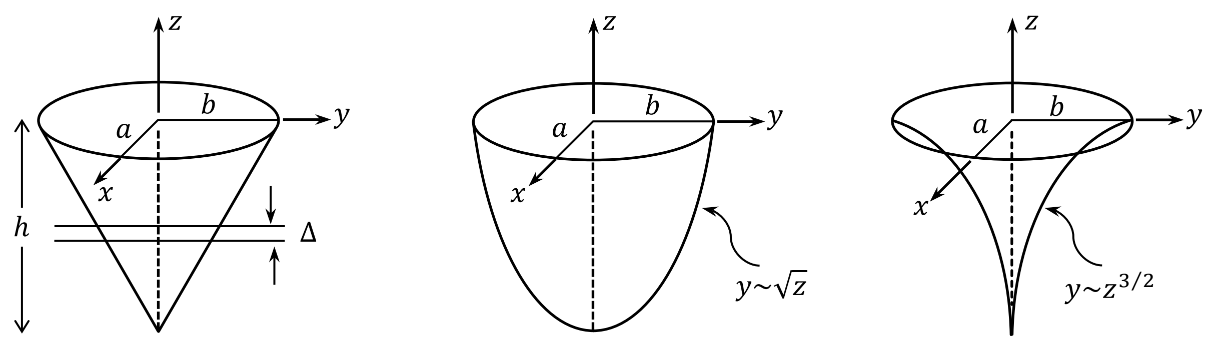
10 pts **Problem 3**) a) The figure on the left-hand side shows a right elliptical cone whose base is the ellipse $(x/a)^2 + (y/b)^2 = 1$, and whose apex is located a distance h from the base on the perpendicular line drawn from the center of the ellipse. Consider a thin slice of the cone at elevation $z = n\Delta$, then add up the volumes of all the slices from $n = 1$ to N, where $N = h/\Delta$ is the total number of such slices. Show that, in the limit when $\Delta \rightarrow 0$, the volume of the cone thus computed approaches the well-known value $V = \pi abh/3$.

Repeat part (a) for a similar geometrical shape whose cross-sectional diameters grow not in proportion to the height z, but (b) in proportion to $\sqrt{z}$, and (c) in proportion to $z^{3/2}$.



Hint: The area of the ellipse at the base is πab;

$1 + 2 + 3 + \cdots + N = N(N+1)/2$;

$1^2 + 2^2 + 3^2 + \cdots + N^2 = N(N+1)(2N+1)/6$;

$1^3 + 2^3 + 3^3 + \cdots + N^3 = [N(N+1)/2]^2$.

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6 pts **Problem 4**) Let x and y represent the length and width of a rectangle. Use the method of Lagrange multipliers to determine the values of x and y that maximize the area of the rectangle subject to the constraint that its perimeter is fixed at P.

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6 pts **Problem 5**) The Taylor series expansion of $f(x) = (x^2 - 2x + 2)^{-1}$ around the point $x_0 = 0$ is written as $\sum_{n=0}^{\infty} a_n x^n$. Use a recursive method (similar to that used to find the Bernoulli numbers) to determine the first 15 coefficients $a_0, a_1, a_2, \cdots, a_{14}$ of the above Taylor series.
