Opti 503A **Final Exam** (5/7/2012) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Problem 1) Let $f(x)$ and $g(x)$ be two arbitrary functions of the real variable x whose Fourier transforms are given by $F(s)$ and $G(s)$, respectively. In general, $f(x)$, $g(x)$, $F(s)$, and $G(s)$ are complex-valued functions of their respective real variables.

4 pts a) Using the defining integrals of direct and inverse Fourier transformation, prove the following identity, which is commonly referred to as Parseval's theorem:

$$\int_{-\infty}^{\infty} f(x)g^*(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} F(s)G^*(s)\,\mathrm{d}s.$$

Considering that $\mathscr{F}\{\mathrm{Rect}(x)\} = \mathrm{sinc}(s)$, $\mathscr{F}\{\mathrm{Tri}(x)\} = \mathrm{sinc}^2(s)$, and $\mathscr{F}\{\exp(-|x|)\} = 2/[1+(2\pi s)^2]$, use Parseval's theorem to evaluate the following definite integrals:

2 pts b) $\int_{-\infty}^{\infty} \mathrm{sinc}^3(s)\,\mathrm{d}s$,

2 pts c) $\int_{-\infty}^{\infty} \mathrm{sinc}^4(s)\,\mathrm{d}s$,

2 pts d) $\int_{0}^{\infty} \exp(-x)\,\mathrm{sinc}(x)\mathrm{d}x$.

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10 pts **Problem 2**) The Bessel function of first kind, order n has the following Taylor series expansion:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,(n+m)!}.$$

Show by direct substitution into the Bessel equation $x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x) = 0$, that $J_n(\alpha x)$ satisfies the Bessel equation. Here α is an arbitrary real-valued constant.

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10 pts **Problem 3**) Use the method of Frobenius to find the general solution to the following linear, ordinary differential equation with constant coefficients:

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + 2\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) = 0.$$

Hint: The indicial equation has three solutions, $s_1 = 0$, $s_2 = 1$, and $s_3 = -1$. While s_1 and s_3 lead to the most general solution of the differential equation, it is easier to start with s_2 in order to obtain one of the two independent solutions. The other solution may then be found using either s_1 or s_3.

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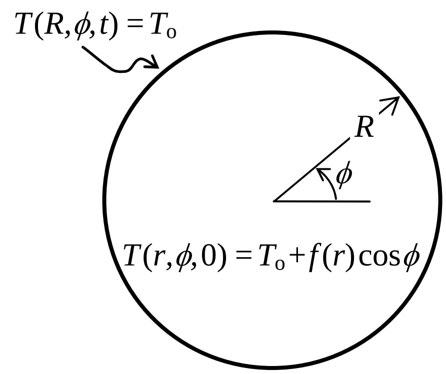
10 pts **Problem 4**) A thin, solid disk of radius R and thermal diffusivity D [cm^2/sec] has an initial temperature distribution $T(r,\phi,t=0) = T_o + f(r)\cos\phi$. Here T_o is the constant ambient temperature, $f(r)$, a function of the radial coordinate r, is specified in the interval $0 \le r \le R$, and the azimuthal angle ϕ covers the entire available range from 0 to 2π. The boundary of the disk at

$r = R$ is kept at the constant ambient temperature at all times $t \geq 0$, so that $T(r = R, \phi, t) = T_o$. (The disk is sufficiently thin, so that its temperature profile through the thickness may be assumed to be uniform.) Obtain the solution to the 2-dimensional heat diffusion equation $D\nabla^2 T(r,\phi,t) = \partial T(r,\phi,t)/\partial t$ for $t \geq 0$ using the method of separation of variables.



Hint: The Laplacian operator in cylindrical coordinates is $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$.