The Electron Oscillator/Lorentz Atom

Consider a simple model of a classical atom, in which the electron is harmonically bound to the nucleus

\[ F_{en} = -m\omega_0^2 x \]

**Note:** We should regard this as a model of the response of an atom, rather than a classical model of the atom itself.

We can justify this response model as follows. Quantum mechanics suggests we can describe the atom as a point-like nucleus and an electron cloud. As a rough approximation we can take the latter to be of uniform density \( \rho \) and radius \( R \), i.e.

\[ e = \frac{4\pi}{3} \rho R^3 \Rightarrow \rho = \frac{3e}{4\pi R^3} \]

We now borrow a well known result for the force on a test charge due to a spherical charge distribution, and find

\[ F_{ne} = -\frac{eq(r)}{4\pi\varepsilon_0} \frac{r}{r^3} = -F_{en} , \]

where \( r \) is the displacement of the nucleus away from the center of the electron cloud, and \( q(r) \) is the charge contained in a sphere of radius \( r \) and centered on the cloud center, i.e. the force is equal to the force arising if the charge \( q(r) \) was located at the cloud center. Electron charge outside radius \( r \) contributes no force. Setting \( r = -x \) we get
\[
\mathbf{F}_{en} = \frac{e}{4\pi\varepsilon_0} \left( \frac{4\pi}{3} r^3 \rho \right) \frac{\mathbf{r}}{r^3} = -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{R^3} \mathbf{x} = -m\omega_0^2 \mathbf{x}, \quad \omega_0 = \sqrt{\frac{e^2/mR^3}{4\pi\varepsilon_0}}.
\]

Thus we obtain a harmonic restoring force when the electron cloud is off center from the nucleus. Furthermore, if we substitute \( R = a_0 \) (one Bohr radius), we get \( \omega_0 \sim 7 \times 10^{15} \) Hz, which is within a factor of two from the 1S-2P transition frequency in Hydrogen!

This analysis goes a long way towards explaining why the electron oscillator model is so remarkably successful. Further insight into the success of the model will be gained when we compare it to the predictions of a quantum mechanical model.

Now substitute the harmonic restoring force \( \mathbf{F}_{en} = -m\omega_0^2 \mathbf{x} \) into the equation of motion for \( \mathbf{x} \) to find

\[
\frac{d^2}{dt^2} \mathbf{x} + \omega_0^2 \mathbf{x} = \frac{e}{m} \mathbf{E}(\mathbf{R},t)
\]

We can combine this with

\[
\mathbf{P} = \mathbf{Np}, \quad \mathbf{p} = e\mathbf{x}
\]

where \( \mathbf{N} \) is the **Number Density** of atoms. This relates the macroscopic quantity \( \mathbf{P} \) to the microscopic quantity \( \mathbf{x} \)

We now have

<table>
<thead>
<tr>
<th>Maxwell’s equations</th>
<th>Maxwell-Lorentz equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorentz model</td>
<td>- We can seek self-consistent solutions to wave propagation</td>
</tr>
</tbody>
</table>

Later in the course we will replace the classical model of the atom with a full quantum model, which will lead to the analogous **Maxwell-Bloch equations**.
Classical Model of Absorption

Maxwell’s equations predict that an oscillating dipole will lose energy by radiating an electromagnetic field. It is therefore necessary to refine our electron oscillator model to include **damping**.

**Note:** We can calculate the energy loss from an isolated dipole entirely within the framework of classical fields and particles. The result is remarkably close to the correct damping rate for Hydrogen, when compared to the predictions of Quantum Electrodynamics (QED).

**Note:** This idea breaks down in media that are perfectly homogeneous on scales above typical interatomic separations, in which case there is no loss as long as the scattering is coherent (has a well defined phase relative to the driving field).

Destructive interference in all directions except forward.

Thus scattering losses occur only if there are fluctuations in the optical properties of the medium on the scale of a wavelength.

**Gasses, liquids and doped solids:** Often homogeneous enough to suppress scattering in non-forward directions.

**Quantum Mechanics:** Incoherent scattering occurs in some regimes.

**The Lorentz Model:** - add an ad hoc friction term w/ $\beta \ll \omega_0$
   (sub- critical damping)

\[
\frac{d^2}{dt^2} x + 2 \beta \frac{d}{dt} x + \omega_0^2 x = \frac{e}{m} E(R,t)
\]

- this is our **basic equation** for atomic response

3
Note: This equation is separable into 3 scalar equations for $x, y, z$. If $\omega_0$ and $\beta$ are the same for the $x, y$ and $z$ directions the medium is isotropic, and $x$ is parallel to $E$.

In crystals the restoring force is not necessarily parallel to the displacement, $F_{en} = -Sx$, where $S$ is a tensor. In that case $x$ and $E$ are not necessarily parallel, and the equation for $x$ is not separable.

In some cases a coordinate system exists in which $S$ is diagonal. In that situation the equation for $x$ separates into three equations with potentially different oscillation frequencies $\omega_x, \omega_y$ and $\omega_z$.

Note: The equation is linear in $x$ and $E$, which implies a superposition principle. This description breaks down in strong fields where we enter the regime of nonlinear optics.

**Solution of the Equation of Motion for $x$:**

From the mathematical theory of differential equations we know that the complete solution can be found as the sum of the general solution without a source term, and one solution with a source term.

Solutions without source terms decay as $e^{-\beta t}$, so the steady state solution equals the solution with a source term.

**Homework:** Solve the equation of motion for $x$ with

Driving field $E(R,t) = \hat{e}E_0 e^{-i(\omega t - kz)}$,

Trial solution $x(R,t) = a e^{-i(\omega t - kz)}$,

where $a$ is a complex amplitude. Show that

$$a = -\hat{e} \frac{(e/m)E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega}$$
Note: We could also switch into the Fourier domain, in which case the equation of motion becomes a simple algebraic equation that relates $x(k, \omega)$ to $E(k, \omega)$.

**Physical quantities:**

\[
\text{Re}[E(R, t)] = \hat{\varepsilon}E_0 \cos(\omega t - kz)
\]

\[
\text{Re}[p(R, t)] = \text{Re}[e\varepsilon(R, t)] = \hat{\varepsilon}E_0 \frac{e^2}{m} \frac{(\omega_0^2 - \omega^2)\cos(\omega t - kz) + 2\beta \omega \sin(\omega t - kz)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}
\]

Note: Here we have assumed that the polarization vector $\hat{\varepsilon}$ is real, i.e. that the electromagnetic field is linearly polarized.

Note: $p$ and $E$ generally oscillate out of phase

\[
\begin{align*}
\omega << \omega_0 & \implies p & \text{ & } E \text{ in-phase} \\
\omega = \omega_0 & \implies p \text{ lags } E \text{ by } \pi/2 \\
\omega >> \omega_0 & \implies p \text{ lags } E \text{ by } \pi
\end{align*}
\]

best to stick with complex notation!

**Complex Polarizability**

We define the complex polarizability $\alpha(\omega)$ as follows

\[
p = e\varepsilon = eae^{-i(\omega t - kz)} = \alpha(\omega)\hat{\varepsilon}E_0e^{-i(\omega t - kz)}
\]

\[
\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta \omega} = \frac{e^2}{m} \frac{\omega_0^2 - \omega^2 + 2i\beta \omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}
\]

It is then easy to show that if $E(r, t) = \hat{\varepsilon}E_0e^{-i(\omega t - kz)}$ and $P = Np$, then the wave equation reduces to
\[
\left(-k^2 + \frac{\omega^2}{c^2}\right)\hat{E}_0 e^{-i(\omega t-kz)} = -\frac{\omega^2}{c^2} \frac{N\alpha(\omega)}{\varepsilon_0} \hat{E}_0 e^{-i(\omega t-kz)}
\]

We thus have plane wave solutions, with a dispersion relation \( k = n(\omega)\omega / c \) that obeys

\[
k^2 = \frac{\omega^2}{c^2} \left[ 1 + \frac{N\alpha(\omega)}{\varepsilon_0} \right] = \frac{\omega^2}{c^2} n^2(\omega)
\]

Here \( n(\omega) \) is the complex index of refraction.

Switching into the Fourier domain, we can introduce the complex susceptibility

\[
P(\omega) = \varepsilon_0 \chi(\omega) E(\omega), \quad \chi(\omega) = \frac{N\alpha(\omega)}{\varepsilon_0}.
\]

Recalling that \( E(\omega) = FT[E(t)] \) and \( P(t) = \varepsilon_0 \int_{-\infty}^{t} R(t-t')E(t')dt' \), and that convolution in the time domain corresponds to multiplication in the frequency domain, we see that the complex susceptibility is related to the linear response function as

\[
\chi(\omega) = FT[R(\tau)].
\]

Thus we have

<table>
<thead>
<tr>
<th>time domain description</th>
<th>frequency domain description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(t), \ R(\tau) )</td>
<td>( FT )</td>
</tr>
<tr>
<td>( FT )</td>
<td>( P(\omega), \ \chi(\omega) )</td>
</tr>
</tbody>
</table>
Complex Index of Refraction

Let

\[ n(\omega) = n_R(\omega) + i n_I(\omega) \]

and consider the propagation of a plane wave

\[ E(z,t) = \hat{E} e^{-i(\omega t - k z)} = \hat{E} e^{-i(\omega t - [n(\omega)\omega/c] z)} = \hat{E} e^{-i(\omega t - n_I(\omega)\omega z/c)} e^{-io(\omega t - n_R(\omega)z/c)} \]

We can now identify

\[ \frac{c}{\omega n_I(\omega)} \quad \text{attenuation length} \]
\[ \frac{c}{n_R(\omega)} \quad \text{phase velocity} \]

Absorption

The intensity of the plane wave electromagnetic field \( E \) is

\[ I_\omega(z) = \frac{1}{2} n_R(\omega) c \varepsilon_0 |E(z,t)|^2 = I_\omega(0) e^{-2n_I(\omega)\omega z/c} = I_0 e^{-a(\omega)z}, \]

where the absorption or extinction coefficient is given by

\[ a(\omega) = 2n_I(\omega)\omega/c = \frac{2\omega}{c} \text{Im} \left[ \left( 1 + \frac{\text{Im} \varepsilon(\omega)}{\varepsilon_0} \right)^{1/2} \right] \]

Possibility of Gain?

We have

\[ n(\omega)^2 = 1 + \frac{Ne^2}{m \varepsilon_0} \left( \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2} + 2i\beta \omega \right) + 4 \beta^2 \omega^2 = n_R^2 - n_I^2 + 2i n_R n_I, \]
and thus
\[ n_Rn_I = \frac{Ne^2}{m\varepsilon_0} \frac{\beta\omega}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2} \geq 0 \]

Now assume the wave propagates in the +z direction with positive phase velocity, \( \text{Re}[k] > 0 \), in which case we must have \( n_R > 0 \). This implies

\[ n_I(\omega) \geq 0 \Rightarrow a(\omega) \geq 0 \]

The fact that the absorption coefficient is always positive rules out the possibility of optical gain and the existence of optical amplifiers and oscillators (lasers), according to the theory of classical linear optics.

**Note:** Physically, gain is impossible because nothing other than the electric field is driving the electron oscillator, and thus there is no source of energy to compensate for energy loss via radiation into the plane wave spatial mode.

**Non-linear response** \( \Rightarrow \) fields at \( \omega_1, \omega_2 \) can create gain at \( \omega_3 = \omega_1 + \omega_2 \).

This is the basis for Optical Parametric Amplifiers (OPA’s) and Oscillators (OPO’s), which are the non-linear optics equivalent of the laser.

We shall see in a following section that the **quantum theory** of atomic response leads to phenomena such as gain and saturation.

**Absorption and Dispersion in Gases**

| \(|\omega_0 - \omega| \ll \omega_0, \omega\) | near-resonance |
| \(|n(\omega)| \sim 1\) | weakly polarizable |

Let
\[ \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega(\omega_0 - \omega) \Rightarrow \]
\[ \alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \approx \frac{e^2/2m\omega}{\omega_0 - \omega - i\beta} = \frac{e^2}{2m\omega} \frac{\omega_0 - \omega + i\beta}{(\omega_0 - \omega)^2 + \beta^2} \]
Furthermore, \[ n(\omega)^2 = 1 + \frac{N\alpha(\omega)}{\epsilon_0} = 1 + \epsilon, \quad \epsilon \text{ small} \]

We expand to first order, \((1 + \epsilon)^{1/2} \approx 1 + \epsilon/2\), to get

\[
\begin{align*}
n_R(\omega) &= 1 + \frac{N\epsilon^2}{4\epsilon_0 m\omega} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \beta^2} \quad \text{dispersive line shape} \\
n_I(\omega) &= \frac{Ne^2}{4\epsilon_0 m\omega} \frac{\beta}{(\omega_0 - \omega)^2 + \beta^2} \quad \text{Lorentzian line shape}
\end{align*}
\]

General behavior:

Note: dispersion \( \propto \frac{1}{\omega_0 - \omega} \) for \(|\omega_0 - \omega| \gg \beta\)

absorption \( \propto \frac{1}{(\omega_0 - \omega)^2} \)

This is the reason we can have loss-less dispersive media.
Glasses and dielectric crystals tend to have strong transitions at UV wavelengths. When considering light propagation in the visible and IR range, the approximations of near-resonance and weak polarizability do not hold, and the above scaling of \( n_R(\omega) \) and \( n_I(\omega) \) in the frequency tails is different. Even so, absorption falls off much faster than dispersion as function of \( \omega_0 - \omega \), and we can have loss-less media with significant dispersion.

Question to class: From the above, we see that

\[
n_R(\omega) < 1 \quad \text{for} \quad \omega > \omega_0 \quad \Rightarrow \quad \text{the phase velocity} \quad \frac{c}{n_R(\omega)} \gg c
\]

How do we interpret this result physically?

**Free Electrons**

Consider the limit \( \omega >> \omega_0 \), corresponding to effectively unbound electrons. This is a reasonable model of plasmas and metals.

We now have

\[
\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \approx - \frac{e^2}{m\omega^2} \Rightarrow \\
n(\omega) = \sqrt{1 + \frac{N\alpha(\omega)}{\varepsilon_0}} \approx \sqrt{1 - \frac{Ne^2}{\varepsilon_0 m\omega^2}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}
\]

Here we have introduced the **plasma frequency**

\[
\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}
\]

Let \( \omega_0 << \omega << \omega_p \quad |\omega_0 - \omega| >> \beta \} \Rightarrow \quad n(\omega) \quad \text{purely imaginary, but no loss!}
We now have

\[ \mathbf{E}(z,t) = \hat{\mathbf{e}} E_0 e^{-i\omega [t - n(\omega) z / c]} = \hat{\mathbf{e}} E_0 e^{-i \omega t} e^{i \sqrt{\omega^2 - \omega_p^2} z / c}, \]

where \( b(\omega) \) is real-valued and positive.

**Note:** This result shows that the wave is not propagating, yet there is no loss. The implication is that a wave traveling through vacuum will be reflected at the boundary of a medium of this type. The penetration depth is \( \approx 1 / b(\omega) \).

Examples of this kind of medium includes plasmas, and metals such as aluminum, silver and gold which are known to be excellent mirrors for visible and IR radiation.

**Fast and Slow Light**

We have seen that the phase velocity of a plane wave in a weakly polarizing medium is given by an expression of the form

\[ v_p = \frac{c}{n_R(\omega)} = \frac{c}{1 + \kappa \frac{\Delta}{\Delta^2 + \beta^2}}, \quad \Delta = \omega_0 - \omega \]

This velocity can be larger than the speed of light. However, the propagation velocity of a pulse is not the phase velocity but rather the group velocity

\[ v_g = \frac{d\omega}{dk} \]

(see e.g. Jackson, Classical Electrodynamics, 3rd Ed., Ch. 7)

Let \( k = n_R(\omega) \frac{\omega}{c} \Rightarrow \frac{d\omega}{dk} = \frac{1}{dk/d\omega} \Rightarrow \]

\[ v_g = \frac{c}{n_R(\omega) + \omega \frac{dn_R(\omega)}{d\omega}} \]
Note: If $n_R(\omega) \sim 1$ and $\omega \left| \frac{dn_R(\omega)}{d\omega} \right| \sim 1$ then \[
\begin{align*}
  v_g &< c \quad \text{slow light} \\
v_g &> c \quad \text{fast light}
\end{align*}
\]

Fast light: According to the electron oscillator/Lorentz atom model, we can have anomalous dispersion, $\frac{dn_R(\omega)}{d\omega} < 0$, and also $\omega \left| \frac{dn_R(\omega)}{d\omega} \right| \sim 1$ near a resonance.

However, in this situation we always have significant absorption, which renders the resulting faster than light pulse propagation much less interesting.

Note: The group velocity is the propagation velocity of a smooth pulse with finite spectral width. This situation is unphysical because such a pulse must have infinite leading and lagging tails. A physical pulse used for communication must have a true “leading edge” where it turns on. Information thus travels at the “edge velocity”, which is always less than $c$.

Slow light: In the electron oscillator model, large values of $n_R(\omega)/d\omega$ occurs only near resonance, have the wrong sign, and are accompanied by strong absorption. This seems to rule out extremely slow light propagation. However, it is possible to overcome this limitation by using a quantum trick involving 3-level atoms.

Signal: weak field at $\omega_s$

Pump: strong field at $\omega_p$

Here, the presence of the strong pump field modifies both the dispersion and absorption of the weak signal field. The resulting behaviors look qualitatively like this:
Thus the signal has an extremely steep dispersion curve and low group velocity at $\omega_s = \omega_p$, along with near-zero absorption. The tradeoff is that these types of Raman resonances tend to be very narrow, which severely limits the pulse bandwidth and duration.
The Dipole Force

We can use the Electron Oscillator model to calculate the force that light exerts upon an atom.

We start from the electric-dipole interaction energy \(-\hat{p} \cdot E\)

Recall that the physically real quantities are

\[
\begin{align*}
E &= \text{Re}[\hat{e}E_0(R)e^{-i\omega t}] \\
\hat{p} &= \text{Re}[\alpha(\omega)\hat{e}E_0(R)e^{-i\omega t}]
\end{align*}
\]

where the polarizability

\[
\alpha(\omega) = \frac{e^2}{2m\omega} \frac{\Delta + i\beta}{\Delta^2 + \beta^2}, \quad \Delta = \omega_0 - \omega.
\]

In that case

\[
V(x, R, t) = -\hat{p} \cdot E = \text{Re}\left[\hat{e} \frac{e^2 E_0(R)}{2m\omega} e^{-i\omega t} \frac{\Delta + i\beta}{\Delta^2 + \beta^2} \text{Re}[\hat{e}E_0(R)e^{-i\omega t}]\right] = -\frac{e^2}{2m\omega} \left(\frac{\Delta}{\Delta^2 + \beta^2} \text{Re}[\hat{e}E_0(R)e^{-i\omega t}]^2 + \frac{\beta}{\Delta^2 + \beta^2} \text{Re}[\hat{e}E_0(R)e^{-i\omega t}]\text{Im}[\hat{e}E_0(R)e^{-i\omega t}]\right)
\]

Averaging over one cycle of the electric field gives us

\[
\langle \text{Re}[\hat{e}E_0(R)e^{-i\omega t}]^2 \rangle = \frac{1}{2} |E_0(R)|^2
\]

\[
\langle \text{Re}[\hat{e}E_0(R)e^{-i\omega t}]\text{Im}[\hat{e}E_0(R)e^{-i\omega t}] \rangle = 0
\]

Substituting this gives us the cycle-averaged light shift potential and dipole force,

\[
V(x, R, t) = -\frac{e^2 |E_0(r)|^2}{4m\omega} \frac{\Delta}{\Delta^2 + \beta^2}
\]

\[
F_{dip}(R) = -\nabla_R V(x, R, t) = \frac{e^2}{4m\omega} \nabla_R |E_0(r)|^2 \frac{\Delta}{\Delta^2 + \beta^2}
\]
Note: For $|\Delta| >> \beta$ we have $V \propto 1/\Delta$. This means we can have a substantial light shift and dipole force without absorption and loss, by tuning far from resonance.