Signal to Noise Calculations

From D* expression:

\[ \text{SNR} = \frac{\phi_e^d(\lambda)D^*(\lambda)}{\sqrt{A_d \Delta f}} \]

\[ \frac{S}{N} = \text{SNR} = \frac{1}{\sqrt{A_d \Delta f}} \cdot \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} d\lambda \phi_e^d(\lambda)D^*(\lambda) \]

Radiant Power on Detector:

- Point Source - sub resolution - \( \frac{1}{\text{Range}^2} \)
- Extended Source - Fills IFOV
Point Sources of Radiant Power

\[ \phi^d_e (\lambda) = \frac{\varepsilon(\lambda)L(\lambda)\tau_a (\lambda)\tau_o (\lambda)A_s A_o}{R^2} \]

Where

\[ \bar{\varepsilon} = \frac{\int d\lambda\varepsilon(\lambda)\tau_a (\lambda)\tau_o (\lambda)D^* (\lambda)L^s_e (\lambda)}{\int d\lambda\tau_a (\lambda)\tau_o (\lambda)D^* (\lambda)L^s_e (\lambda)} \]

\[ \bar{\tau}_a = \frac{\int d\lambda\tau_a (\lambda)\tau_o (\lambda)D^* (\lambda)L^s_e (\lambda)}{\int d\lambda\tau_o (\lambda)D^* (\lambda)L^s_e (\lambda)} \]

\[ \bar{\tau}_o = \frac{\int d\lambda\tau_o (\lambda)D^* (\lambda)L^s_e (\lambda)}{\int d\lambda D^* (\lambda)L^s_e (\lambda)} \]

\[ \text{SNR} = \frac{A_s A_o}{R^2 \sqrt{A_d \Delta f}} \cdot \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} d\lambda \cdot \bar{\varepsilon}L^s_e (\lambda) \bar{\tau}_a \bar{\tau}_o D^* (\lambda) \]
$$\text{SNR} = \frac{A_s A_o \bar{\varepsilon} \tau_a \tau_o}{R^2 \sqrt{A_d \Delta f}} \int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda L^{BB}(\lambda) D^*(\lambda)$$

Design Optimization: 
\(A_s \varepsilon \uparrow \quad R \downarrow\)

\(A_o \uparrow \quad A_d \downarrow\)

If BLIP, then optimum detector radius is about 0.6 of the radius to 1st zero of Bessel Function (airy disc).
SNR for BLIP D* for Point Source

Using:

\[ D^* (\lambda) = \frac{\lambda}{\lambda_{\text{max}}} \cdot D^*_{\text{peak}} \]

\[ L^B_{e} (\lambda) = L^B_{q} (\lambda) \cdot \frac{hc}{\lambda} \]

\[ D^*_{\text{peak}} = \frac{\lambda_{\text{peak}}}{hc} \sqrt{\frac{\eta}{kE^B_{q}}} \]

\( k = 2 \) or \( 4 \) depending on PV or PC

Substitute in the SNR:

\[ SNR = \frac{A_s A_o \overline{\epsilon \tau_a \tau_o}}{R^2 \sqrt{A_d \Delta f}} \cdot \sqrt{\frac{\eta}{k}} \cdot \frac{\int d\lambda L^s_{e}}{\int d\lambda E^B_{q}} \]

To maximize signal to noise ratio:

\[ \eta \uparrow, E^B_{q} \downarrow \]
Extended Source SNR

Recall: \[ \frac{A_s}{R_2} = \frac{A_d}{f^2} \] and \[ \phi_d = \frac{L_s^e A_s A_0}{R^2} = \frac{L_s^e A_d A_0}{f^2} \] ← Same as if at lens

Therefore: \[ \text{SNR} = \frac{A_d}{f^2 \sqrt{A_d \Delta f}} \bar{\varepsilon} \tau_a \tau_0 \int d\lambda L^e_s(\lambda) D^*(\lambda) \] ← Non-BLIP \( D^* \)

\[ = \frac{\pi}{4} \left( \frac{D_0}{f} \right) \frac{D_0 \sqrt{A_d}}{f} \cdot \bar{\varepsilon} \tau_a \tau_0 \int d\lambda L^e_s D^*(\lambda) \]

Assume detector collects energy to first zero of diffraction pattern which is 0.84 of energy. \[ \frac{D_0 \sqrt{A_d}}{f} = 2.44 \lambda \]

Recall: \[ D_{\text{max}}^*(\lambda) = \frac{\lambda_{\text{max}}}{hc} \sqrt{\frac{\eta}{k L_q^{BG}}} \] and \[ E_q^{BG} = \frac{\pi L_q^{BG}}{4 (F/#)^2} \]

\[ D_{\text{max}}^*(\lambda) = \frac{\lambda_{\text{max}}}{hc} \cdot \frac{2(F/#)}{\sqrt{\pi}} \sqrt{\frac{\eta}{k L_q^{BK}}}. \]

Lesson 13-5
Substitute into SNR

\[
SNR = 0.84(2.44\lambda) \frac{\sqrt{\pi} \, \bar{\epsilon} \tau_a \tau_o}{2 \sqrt{\Delta f}} \cdot \sqrt{\frac{\eta}{k}} \cdot \int d\lambda L_{e}^{BB} \sqrt{d\lambda L_{q}^{BB}}
\]

Extended source, BLIP operated

If all efficiencies are unity:

\[
\bar{\epsilon} = \bar{\tau}_a = \bar{\tau}_0 = 1
\]

\[
SNR = 1.81 \frac{\lambda}{\sqrt{k\Delta f}} \frac{\int d\lambda L_{e}^{s}(\lambda)}{\int d\lambda L_{q}^{BG}(\lambda)}
\]
Irradiance on Focal Plane Due to Optics and Extended Scene

\[ E_{\text{scene}} = \varepsilon_{sc} L_e^{\text{scene}} (T_{sc}) \tau_a \tau_o \Omega_s \]

\[ E_{\text{syst}} = L_{\text{syst}}^{BB} (T_{\text{sys}}) \Omega \left[ \varepsilon_1 + \tau_1 \varepsilon_2 + \tau_1 \tau_2 \varepsilon_3 + \ldots \right] \]

\[ \frac{E_{\text{syst}}}{E_{\text{scene}}} = \frac{L_{\text{syst}}^{BB} (T_{\text{sys}}) \varepsilon_o (1 + \tau + \tau^2 + \tau^3 + \ldots)}{L_{\text{scene}}^{BB} (T_{\text{scene}}) \varepsilon_{sc} \tau^m \tau_a} \]

For example, \( \varepsilon_o = 0.03, \varepsilon_{sc} = 0.9, \tau = 0.9, \tau_a = 0.8 \), for three elements (m = 3)

\[ \frac{E_{\text{syst}}}{E_{\text{scene}}} = .015 \frac{L_{\text{syst}}^{BB} (T_{\text{sys}})}{L_{\text{scene}}^{BB} (T_{\text{scene}})} \]
Noise-Equivalent Temperature Difference (NETD)

The change in signal due to a $\Delta T$ just equal to the sensor noise, $N$, is

\[ S(T + \Delta T) - S(T) = N \]

or

\[ \frac{S(T + \Delta T) - S(T)}{N} = \frac{1}{\Delta T} = \frac{1}{NETD} \]

\[ LHS = \frac{\partial SNR}{\partial T} \]

Thus

\[ NETD = \left[ \frac{\partial SNR}{\partial T} \right]^{-1} \]
Finally:

\[
NETD = \left[ 1.81 \frac{\bar{\lambda} \bar{\varepsilon} \bar{\tau}_a \bar{\tau}_o}{\sqrt{\Delta f}} \sqrt{\frac{\eta}{2F}} \sqrt{\int d\lambda \frac{\partial L^{BB}_q}{\partial T}} \int d\lambda L^{BG}_q \right]^{-1}
\]
NETD Measurements and Predictions

• Set up 4-bar test target with different temperature between bars and background of 20 mK

• Measure mean signal over target region

• Measure noise over background region
LWIR Camera Image of 20 mK Bar Target
Data Collected/NETD Measured

- Signal spatial average
  \[ \bar{S} = \frac{1}{M \cdot N} \sum_{m=1}^{M} \sum_{n=1}^{N} P_{m,n} \]

- Noise spatial standard deviation

- For test target shown with 20mK temperature difference

\[
\text{Noise}^2 = \sum_{jk} (P_{jk} - \bar{N})^2 \\
\bar{N} = \frac{1}{J \cdot K} \sum_{j=1}^{J} \sum_{k=1}^{K} P_{j,k}
\]

\[ \bar{S} = 0.817 \quad \text{Noise} = 0.435 \]

\[ \text{NETD} = \frac{\Delta T}{\bar{S}/\text{noise}} = \frac{0.02}{0.817/0.435} = 0.0106 \text{K} \]
NETD Predicted

Parameters of syst
Frame rate; 42 = FPS

Duty cycle; 0.03 = \(d_w\)

\[
\Delta f = \frac{1}{2T} = \frac{\text{FPS}}{2d_w}
\]

\(\lambda_1 = 8\mu\text{m}\)

\(\lambda_2 = 11\mu\text{m}\)

\(F/# = 4\)

\(D^* = 2(10^{10})\)

\(\eta = 0.60\)

\(A_d = 61\mu\text{m} \cdot 61\mu\text{m}\)

\(T_{\text{BKg}} = 300\text{K}\)

\[
\text{NETD} = \frac{2\sqrt{2(F/#)}}{\sqrt{\pi\eta A_d}} \cdot \frac{L_{\text{BKg}} \cdot \Delta f}{\left[\int_{\lambda_1}^{\lambda_2} \frac{\partial L_q(\lambda, 300)\,d\lambda}{\partial T}\right]^{-1}}
\]

Predicted: \(\text{NETD} = 0.0056 \text{ K}\)
\[ SNR = \frac{\Phi_e(\lambda) D^*(\lambda)}{\sqrt{A_d A_f}} \]

\[ SNR = \frac{1}{\sqrt{A_d A_f}} \int_{\lambda_{min}}^{\lambda_{max}} \Phi_e(\lambda) D(\lambda) d\lambda \]

\[ \Phi_e(\lambda) = \frac{I_e(\lambda)}{R^2} A_{ent} E \bar{T}_a \bar{T}_o \]

\[ I_e(\lambda) \text{ - spectral Intensity} \]

\[ SNR = \frac{\bar{E} \bar{T}_a \bar{T}_o A_{ent}}{\sqrt{A_d A_f}} \int_{\lambda_{min}}^{\lambda_{max}} I_e(\lambda) D^*(\lambda) d\lambda \]

Rearranging into the range expression:

\[ R^2 = \frac{\bar{E} \bar{T}_a \bar{T}_o A_{ent}}{\sqrt{A_d A_f}} \int_{\lambda_{min}}^{\lambda_{max}} I_e(\lambda) D^*(\lambda) d\lambda \]

\[ SNR \]

This is the basic eq. for a search system.

Lesson 13-14
\[ R = \left[ \frac{\varepsilon T_{0} T_{e} \int I(\lambda) D^{*}(\lambda) d\lambda}{SNR \sqrt{\Delta f}} \cdot \frac{A_{enp}}{\Gamma A_{d}} \right]^{1/2} \]

\[ = \left[ \frac{\varepsilon T_{0} T_{e} \int I(\lambda) D^{*}(\lambda) d\lambda}{SNR \sqrt{\Delta f}} \cdot \frac{\pi D_{e}^{2}}{4 \cdot 1.52 \cdot f} \right]^{1/2} \]

\[ R = \left[ \frac{\varepsilon T_{0} T_{e} \int I_{e}(\lambda) D^{*}(\lambda) d\lambda}{SNR \sqrt{\Delta f}} \cdot \frac{\pi D_{e}}{4 \cdot (F/\#)/(\#)} \right]^{1/2} \]
Range Eq cast in terms of subsystems:

\[ R = \sqrt{\left( \frac{\pi}{4} \cdot \frac{D_{LT}}{F/\#} \right)^2 \left[ \int E_\lambda(x) \cdot D^*(x) dx \right]^2 \cdot \left[ \frac{T_a}{SNR(\Delta f \cdot \Delta t)} \right]^{1/2} } \]

Optics \quad \uparrow \quad \text{Target detector} \quad \uparrow \quad \text{Signal processing}

Obvious, \( D_{LT} \uparrow, F/\# \downarrow, D^* \uparrow, T_a \uparrow \)

Two considerations:

\[ \left( T a \Delta f \Delta t \right)^{-1/2} \quad 4^{th} \text{ root dependence, decrease } \Delta f \Delta t. \]

dwell Time \( \uparrow \), \( \Delta f \downarrow \),

\# of det \( \uparrow \), Frame Time is same, \( \Delta f \downarrow \)

Range \( \propto \left[ N \right]^{1/4} \)

See example prob. in Book pg 500
SNR required.
How does one set this?

Proba of detection and False Alarms
Allowable!

Proba of det ≈ 90%
and 1 FAR of one per day

\[
\int_{V_t}^{\infty} P\left(\frac{V-V_s}{\sqrt{N_0}}\right) dV
\]

\[
\int_{V_t}^{\infty} P\left(\frac{V}{N_0}\right) dV = \text{False Alarms}
\]
\[
\text{Proba of detection} = \int_{V_t}^{\infty} P\left(\frac{V-V_s}{N_0}\right) dV
\]

How you find required SNR.

1. Calc. FAR requirement or \( V_t \) value.

2. Calc. the signal level to have 90% proba of detection

Once done, produce a ROC curve to get Fig. of Merit of system.
A space ship has happy hour. An astronaut has both gin and vermouth but his ice cube ($A_T = 1$ in, $T = 273^\circ$) is lost in space. He has an infrared sensor at his disposal to find it. The sensor characteristics are:

\[
\begin{align*}
D_o &= 1 \text{ m - optical collector diameter} \\
A_s &= 1 \text{ in}^2 \\
\tau_o &= 0.1 - \text{ optical transmission} \\
F/\# &= 1 \\
A_d &= 1.0 \text{mm}^2 - \text{ detector area - bolometer} \\
D^* &= 10^9 \text{ Jones – } \mu \text{ Bolometer}
\end{align*}
\]

What is the maximum range that the ice cube can be from the space ship in order to detect with 90% probability at detection and one false alarm during the hour?
Ice cube 273°K

\[ L_s = \sigma T^4 \]

\[ = 5.65(10^{-12})(273)^4 \cdot \frac{4}{\pi} \]

\[ = 10^{-2} \frac{w}{cm^2 - sr} \]

Radiant power received on entrance pupil for a given range (R):

\[ \phi_e^{ent} = \frac{L_s A_s A_c}{R^2} \]
The astronaut must scan the entire sphere (4\pi steradians) in one hour.

The Dwell Time per IFOV (t_d) to accomplish 4\pi scanning is:

\[
t_d = \frac{10^{-6}}{4\pi} \cdot 3600 \text{ sec} = .286 \text{ ms}
\]
\[
\Delta f = \frac{1}{2t_{\text{d}}} \approx 1748 \text{ Hz}
\]
Point Source Detection at the Detector Plane

\[ D^* = \frac{\sqrt{A_d \Delta f}}{\phi_e^d} \cdot \frac{S}{N} \]

\[ \phi_e^d = \phi_{e \tau}^\text{ent} \]

\[ \phi_{e \tau}^\text{ent} = \frac{\sqrt{A_d \Delta f}}{D^* \cdot \tau} \cdot \frac{S}{N} \]

Also should equal, as previously shown, \( \phi_{e \tau}^\text{ent} \)

\[ \phi_{e \tau}^\text{ent} = \frac{L_s A_s A_c}{R^2} \]
Point Source

- What S/N is required to detect with 90% probability and only one false alarm in one hour?

Two Hypothesis:

$M_0$ - Output voltage with no ice cube in IFOV, only cold space ($3^\circ K$).

$M_1$ - Signal voltage with ice cube in IFOV (source at $273^\circ K$).
Receiver Operating Characteristic (ROC) Curve

- Problem of detection: presence or absence of target

- EXAMPLE 1: observer or algorithm always reports presence of target,
  - 100% Probability of Detection,
  - High false-alarm rate (FAR)

- EXAMPLE 2: observer or algorithm always reports absence of target,
  - No false alarms
  - No targets detected either

- How to quantify these extremes and situations inbetween? First, develop a DECISION MATRIX:

<table>
<thead>
<tr>
<th>Does the observer say the object is PRESENT?</th>
<th>True Positive (TP)</th>
<th>False Positive (FP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>False Negative (FN)</td>
<td>True Negative (TN)</td>
</tr>
</tbody>
</table>

| Is the object really PRESENT? |
|-------------------------------|-------------------|
| Y                             | N                 |

Lesson 13-24
Receiver Operating Characteristic Curve

- $N_{tot}$ measurements,

- Define four fractions:

$$N_{tot} = N_{TP} + N_{TN} + N_{FP} + N_{FN}$$

$$TPF = \frac{N_{TP}}{N_{TP} + N_{FN}}$$, a.k.a. SENSITIVITY

$$TNF = \frac{N_{TN}}{N_{TN} + N_{FP}}$$, a.k.a. SPECIFICITY

$$FPF = \frac{N_{FP}}{N_{TN} + N_{FP}}$$

$$FNF = \frac{N_{FN}}{N_{TP} + N_{FN}}$$

By varying the decision threshold, an ROC curve is generated.

Receiver Operating Characteristic Curve

- ROC curves can be used to compare algorithm or sensor performance
- Ideal case: Curve hugs left and top edges of plot
  - no false positives
  - curve IV
- Worst case: Sensor provides no information and observer has to guess
  - dashed diagonal line
- Area under curve indicates usefulness

Assume Gaussian Distribution of noise \((\sigma)\) being the same for each case \((M_o\ or\ M_1)\).

\[
P(V/M) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(V-M)^2/2\sigma^2}
\]

General form of conditional probability.

\(V_t\) = threshold signal level or decision point. Every signal voltage above \(v_t\) is called an ice cube.
Neyman-Pearson Criterion*:

“Maximize Probability of Detection ($P_d$) while maintaining the probability of false alarm at most at the specified level of false alarm rate - $P_f$.”

APPROACH:

1) Solve for threshold level ($V_t$) to control false alarm rate.

2) Solve for SNR ($M_1$) for 90% probability of detection.

The threshold voltage level, which determines if the ice cube is present:

**Probability of false alarm** for chosen threshold voltage, $V_t$

$$P_f = \int_{V_t}^{\infty} P(V / M_o) dV$$

**Probability of detection** for chosen threshold voltage, $V_t$

$$P_d = \int_{V_t}^{\infty} P(V / M_1) dV$$

Now, false alarm rate $(1/4\pi(10^6))$ which determine the threshold voltage:

$$\frac{1}{4\pi(10^6)} = \int_{V_t}^{\infty} 1 P(V / M_0) dV = \int_{V_t}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(V - M_o)^2}{2\sigma^2}} dV$$

Change of variable:

$$u = \frac{V - M_o}{\sigma}$$

$$= \int_{\frac{V_t - M_o}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2 / 2} du$$
Solving for $V_t$ for the given false alarm rate:

$$\frac{1}{(2\pi)^{0.5}} \cdot \int_{V_t-M_o}^{20} \exp\left(-\frac{u^2}{2}\right) du = \frac{1}{4\pi(10^6)}$$

$$\frac{V_t - M_o}{\sigma} = 5.5$$

This would determine the voltage to set for threshold (is 5.5 std dev) if the mean was zero ($M_o = 0$).

Now we can use the 90% probability of detection required to get the signal to noise ($M_1$) needed.
\[ P_d = 0.9 = \int_{V_t-M_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \]

There is a trick to setting integration limit

**NOTE:**

\[
\frac{V_t-M_1}{\sigma} = \frac{V_t-M_0+M_0-M_1}{\sigma} = \frac{V_t-M_0}{\sigma} - \frac{M_1-M_0}{\sigma} \]

This is SNR for \( P_d = 0.9 \)

\[ P_d = 0.9 = \int_{V_t-M_0-SNR}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \]
Solving for SNR

Given:

\[ \frac{1}{\sqrt{2\pi}} \cdot \int_{5.5 - \text{SNR}}^{\infty} \exp\left(-\frac{u^2}{2}\right) \, du = 0.9 \]

Solving for SNR:

SNR = 6.77

This is the required SNR for determining the range that an astronaut can detect the ice cube.
Calculating the Radiant Power on the Entrance Pupil for an SNR = 6.77

\[ \phi_{e}^{\text{ent}} = \sqrt{\frac{A_d \Delta f (SNR)}{D^*, \tau_o}} \]

\[ \phi_{e}^{\text{ent}} = \frac{\sqrt{10^{-2} \cdot 1748 \cdot 6.77}}{10^9 \cdot .1} \]

\[ \phi_{e}^{\text{ent}} = 3.1(10^{-7}) \text{ watts} \]

Now using the equation for range:

\[ R^2 = \frac{L A_s A_c}{\phi_{e}^{\text{ent}}} \cdot \frac{10^{-2} \cdot 6.45 \cdot \pi(100^2)}{3.1(10^{-7}) \cdot 4} \]

\[ R^2 = 2.0(10^9) \text{ cm} \]

\[ R = 491 \text{ meters} \]